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László Fehér, Géza Kós, Árpád Tóth

MATHEMATICAL ANALYSIS – PROBLEMS AND EXERCISES II

Eötvös Loránd University
Faculty of Science

Typotex

2014

Editors: Géza Kós, Zoltán Szentmiklóssy
Reader: Péter Pál Pach

ISBN 978 963 279 420 4
Prepared under the editorship of Typotex Publishing House
(http://www.typotex.hu)
Responsible manager: Zsuzsa Votisky
Technical editor: József Gerner

Made within the framework of the project Nr. TÁMOP-4.1.2-08/2/A/KMR-2009-0045, entitled "Jegyzetek és példatárak a matematika egyetemi oktatásához" (Lecture Notes and Workbooks for Teaching Undergraduate Mathematics).

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MAGYARORSZÁG MEGÚJUL

A projekt az Európai Unió támogatásával, az Európai Regionális Fejlesztési Alap társfinanszírozásával valósul meg.

KEY WORDS: Analysis, calculus, derivate, integral, multivariable, complex.

SUMMARY: This problem book is for students learning mathematical calculus and analysis. The main task of it to introduce the derivate and integral calculus and their applications.

Contents

Preface

This collection contains a selection from the body of exercises that have been used in problem session classes at ELTE TTK in the past few decades. These classes include the current analysis courses in the Mathematics BSc programs as well as previous offerings of Analysis I-IV and Complex Functions.

We recommend these exercises for the participants and teachers of the Mathematician, Applied Mathematician programs and for the more experienced participants of the Teacher of Mathematics program.

All exercises are labelled by a number referring to its difficulty. This number roughly means the possible position of the problem in an exam. For the Teacher program the range is 1-7, for the Applied Mathematician program 2-8, and for the Mathematician program 3-9. (Usually the students need to solve five problems correctly for maximum grade; the sixth and seventh problems are to challenge the best students.) Problems with difficulty 10 are not expected to appear on an exam, they are recommended for students aspiring to become researchers.

For many exercises we are not aware of the exact origin. They are passed on by “word of mouth” from teacher to teacher, or many times from the teacher of the teacher to the teacher. Many exercises may have been created several generations before.

However one of the sources can be identified, it is “the mimeo”, a widely circulated set of problems duplicated by a mimeograph in the 70’s. The problems within “the mimeo” were mainly collected or created by Miklós Laczkovich, László Lempert and Lajos Pósa.

Let us give only a (most likely not complete) list of our colleagues who were recently giving lectures or leading problem sessions at the Department of Analysis in Real and Complex Analysis:

Mátyás Bognár, Zoltán Buczolich, Ákos Császár, Márton Elekes, Margit Gémes, Gábor Halász, Tamás Keleti, Miklós Laczkovich, György Petruska, Szilárd Révész, Richárd Rimányi, István Sigray, Miklós Simonovics, Zoltán Szentmiklóssy, Róbert Szőke, András Szűcs, Vera T. Sós.

Some problems from the textbook Analízis I. of Miklós Laczkovich and Vera T. Sós are reproduced in this volume with their kind permission. We are grateful for their generosity.

We thank everyone whose help was invaluable in creating this volume, the above mentioned professors and all the students who participated in these classes. As usual when typesetting the problems we may have added some errors of mathematical or typographical nature; for which we take sole responsibility.

Part I

Problems

Chapter 1

Basic notions. Axioms of the real numbers

1.0.1 Fundaments of Logic

1.0.1. (1) Calculate the truth table

$$A \vee (B \Longrightarrow A)$$

Answer→

1.0.2. (3) Calculate the truth tables.
1. $A \Rightarrow B$ 2. $\overline{A \Rightarrow B}$ 3. $A \Rightarrow (B \Rightarrow C)$

1.0.3. (2) Let $P(x)$ mean „x is even” and let $H(x)$ mean „x is divisible by six”. What is the meaning of the following formulas and are they true? ($\neg$ denotes the negation.)

1. $P(4) \wedge H(12)$
2. $\forall x \; \big(P(x) \Rightarrow H(x)\big)$
3. $\exists x \; \big(H(x) \Rightarrow \neg P(x)\big)$
4. $\exists x \; \big(P(x) \wedge H(x)\big)$
5. $\exists x \; \big(P(x) \wedge H(x+1)\big)$
6. $\forall x \; \big(H(x) \Rightarrow P(x)\big)$
7. $\forall x \; \big(\neg H(x) \Rightarrow \neg P(x)\big)$

1.0.4. (3) Let $H \subseteq \mathbb{R}$ be a subset. Formalize the following statements and their negations. Is there a set with the given property?

1. H has at most 3 elements.
2. H has no least element.
3. Between any two elements of H there is a third one in H.
4. For any real number there is a greater one in H.

Answer→

1.0.5. (2) Formalize the statements: 'There is no greatest natural number' and 'There is a greatest natural number' (logical signs, $=$ and $<$ can be used).

1.0.6. (5) What is the meaning of the following statements if $H \subset \mathbb{N}$?
(a) $(1 \in H) \wedge (\forall x \in H\ (x+1) \in H)$;
(b) $(1 \in H) \wedge (2 \in H) \wedge (\forall x \in \mathbb{N}\ (x \in H \wedge (x+1) \in H) \Rightarrow (x+2) \in H)$;
(c) $(1 \in H) \wedge ((\forall x \in \mathbb{N}\ (\forall y \in \mathbb{N}\ y < x \Rightarrow y \in H)) \Rightarrow x \in H)$;
(d) $\forall x \in \mathbb{N}\ (x \notin H) \Rightarrow (\exists y \in N\ (y < x \wedge y \notin H)$;

1.0.7. (7) How many sets $H \subset \{1, 2, \dots, n\}$ do exist for which $\forall x\ (x \in H \implies x+1 \notin H)$?

1.0.8. (7) How many sets $H \subset \{1, 2, \dots, n\}$ do exist for which $\forall x\ ([(x \in H) \wedge (x+1 \in H)] \Rightarrow x{+}2 \in H)$?

Hint→

1.0.9. (5) Which statement does imply which one?

1. $(\forall x \in H)(\exists y \in H)(x+y \in A \wedge x-y \in A)$;
2. $(\exists x \in H)(\forall y \in H)(x+y \in A \wedge x-y \in A)$;
3. $(\forall x \in H)(\exists y \in H)(x+y \in A)$.

1.0.10. (4) What is the meaning of the following formulas if H is a set of numbers?
(a) $\forall x \in \mathbb{R}\ \exists y \in H\ x < y$; (b) $\forall x \in H\ \exists y \in \mathbb{R}\ x < y$; (c) $\forall x \in H\ \exists y \in H\ x < y$.

1.0.11. (5) Let A and B two sets of numbers, which statement implies which one?

(a) $\forall x \in A\ \exists y \in B\ x < y$ (c) $\forall x \in A\ \forall y \in B\ x < y$

(b) $\exists y \in B\ \forall x \in A\ x < y$ (d) $\exists x \in A\ \exists y \in B\ x < y$

1.0.12. (5) Prove that the implication is left distributive with respect to the disjunction.

Solution→

Related problem: 1.0.13

1.0.13. (5) (a) Is it true that the implication is right distributive with respect to the conjunction?

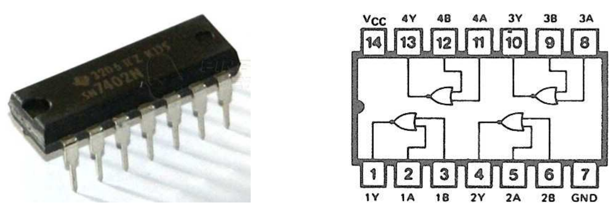
(b) Is it true that the implication is left distributive with respect to the conjunction?

Related problem: 1.0.12

1.0.14. (4) Let $\mathrm{NOR}(x,y) = \neg(x \vee y)$. Using only the NOR operation we can create several expressions, e.g., $\mathrm{NOR}(x, \mathrm{NOR}(\mathrm{NOR}(x,y), \mathrm{NOR}(z,x)))$.

(a) Show that we can generate all logic functions of n variables in this way!

(b) Show another example of a logic function of 2-variable NOR with this generating property!



A Texas Instruments SN7402N integrated circuit, with 4 independent NOR logic gates

Hint→

1.0.15. (6) Show that any Boolean function $f(x_1, x_2, \ldots, x_n)$ of n variables (i.e. a function assigning a true/false value to n true/false values) can be expressed by using only variable names, brackets, the constant false value and the implication operation ($\Rightarrow$).

1.0.16. (8) Show that a Boolean function $f(x_1, x_2, \ldots, x_n)$ of n variables (i.e. a function assigning a true/false value to n true/false values) can be expressed by using only variable names, brackets and the implication operation ($\Rightarrow$) if and only if

$$\exists k \in \{1, 2, \ldots, n\} \Big(\forall x_1, \ldots, x_n \; \big(x_k \Rightarrow f(x_1, x_2, \ldots, x_n)\big)\Big).$$

1.0.2 Sets, Functions, Combinatorics

1.0.17. (2) Solve: $|2x - 1| < |x^2 - 4|$.

1.0.18. (3) Find the parallelogram with greatest area with given perimeter.

1.0.19. (2) What are the solutions of the following equation?

$$\left(\frac{x + |x|}{2}\right)^2 + \left(\frac{x - |x|}{2}\right)^2 = x^2$$

1.0.20. (1)

1. How many words of length k can be created using the letters A, B, C, D, E, F, G?
2. How many such word of length 7 can be created without repeating a letter?
3. How many such word of length 7 can be created with the property that A and B are neighbors (no repetition)?

1.0.21. (2) Show that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

1.0.22. (4) Prove the so-called *binomial theorem*:

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n.$$

Hint→

1.0.23. (3) Which one is bigger? 639^9 or $638^9 + 9 \cdot 638^8$?

Hint→

1.0.24. (3) Prove the De Morgan identities, i.e., $\overline{A \cup B} = \overline{A} \cap \overline{B}$, and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

1.0.25. (3) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

1.0.26. (2) Let $A = \{1, 2, ..., n\}$ and $B = \{1, ..., k\}$.

1. How many different functions $f : A \to B$ do exist?
2. How many different injective functions $f : A \to B$ do exist?
3. How many different functions $f : A_0 \to B$ do exist, where $A_0 \subset A$ is arbitrary?

Answer→

1.0.27. (4) Prove that $x \in A_1 \Delta A_2 \Delta \cdots \Delta A_n$ if and only if x is an element of an odd number of A_i's.

1.0.28. (3) Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference of the sets A and B. Show that for any sets A, B, C:
1. $A\Delta\emptyset = A$, 2. $A\Delta A = \emptyset$, 3. $(A\Delta B)\Delta C = A\Delta(B\Delta C)$.

1.0.29. (2) How many lines are determined by n points in the plane? And how many planes are determined by n points in the space?

1.0.30. (3) How many ways can one put on the chessboard:

1. 2 white rooks,
2. 2 white rooks such that they cannot capture each other,
3. 1 white rook and 1 black rook,
4. 1 white rook and 1 black rook such that they cannot capture each other?

1.0.31. (4) How many different rectangles can be seen on the chessboard?

1.0.32. (3) Is it true for all triples A, B, C of sets that
(a) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$;
(b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$;
(c) $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$?

Answer→

1.0.33. (4) Is it true that the subsets of a set H form a ring with identity using the symmetric difference and a) the intersection b) the union?

1.0.34. (4) Let $f : A \to B$. For any set $X \subset A$ let $f(X) = \{f(x) : x \in X\}$ (the *image* of the set X), and for any set $Y \subset B$ let $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$ (the *preimage* of the set Y). Is it true that
(a) $\forall X, Y \in \mathcal{P}(A)$ $f(X) \cup f(Y) = f(X \cup Y)$?
(b) $\forall X, Y \in \mathcal{P}(B)$ $f^{-1}(X) \cup f^{-1}(Y) = f^{-1}(X \cup Y)$?

1.0.35. (4) Let $f : A \to B$. Is it true that
(a) $\forall X, Y \in \mathcal{P}(A)$ $f(X) \cap f(Y) = f(X \cap Y)$?
(b) $\forall X, Y \in \mathcal{P}(B)$ $f^{-1}(X) \cap f^{-1}(Y) = f^{-1}(X \cap Y)$?

1.0.36. (8) Let $A_1, A_2, \dots$ be non-empty finite sets, and for all positive integer n let f_n be a map from A_{n+1} to A_n. Prove that there exists an infinite sequence $x_1, x_2, \dots$ such that for all n the conditions $x_n \in A_n$ and $f_n(x_{n+1}) = x_n$ hold (König's lemma).

1.0.37. (8) Using König's lemma (see exercise 1.0.36) verify that if all finite subgraphs of a countable graph can be embedded into the plane, then the whole graph can be embedded into the plane as well.

1.0.38. (7) Show an example of an associative operation $\circ : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ for which the union operation is left distributive but not right distributive. (Here $\mathcal{P}(\mathbb{R})$ denotes the set of all subsets of the real line $\mathbb{R}$.)

1.0.3 Proving Techniques: Proof by Contradiction, Induction

1.0.39. (7) We cut two diagonally opposite corner squares of a chessboard. Can we cover the rest with 1×2 dominoes? And for the $n \times k$ "chessboard"?

1.0.40. (7) Consider the set $H := \{2, 3, \dots n+1\}$. Prove that

$$\sum_{\emptyset \neq S \subset H} \prod_{i \in S} \frac{1}{i} = n/2.$$

(For example for $n = 3$ we have $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 4} + \frac{1}{2\cdot 3\cdot 4} = \frac{3}{2}$.)

1.0.41. (6) We cut a corner square of a 2^n by 2^n chessboard. Prove that the rest can be covered with disjoint L-shaped dominoes consisting of 3 squares.

1.0.42. (3) Prove that

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\dots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Solution→

1.0.43. (4)

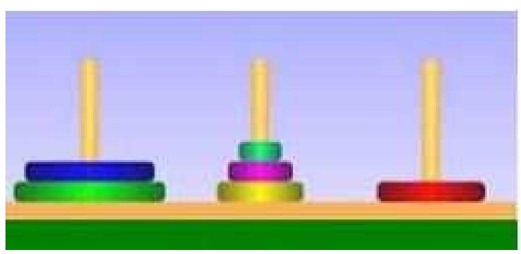
1. Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 3}$. Prove that $\forall n \in \mathbb{N}\ a_n \leq a_{n+1}$.

2. Let $a_1 = 0.9$ and $a_{n+1} = a_n - a_n^2$. Prove that $\forall n \in \mathbb{N}\ a_{n+1} < a_n$ and $0 < a_n < 1$.

1.0.44. (7) Prove that $\tan 1^o$ is irrational!

Hint→

1.0.45. (5) At least how many steps do you need to move the 64 stories high Hanoi tower?



Towers of Hanoi

Hint→

1.0.46. (5) For how many parts the plane is divided by n lines if no 3 of them are concurrent?

1.0.47. (8) For how many parts the space is divided by n planes if no 4 of them have a common point and no 3 of them have a common line?

Hint→

1.0.48. (5) Prove that finitely many lines or circles divide the plane into domains which can be colored with two colors such that no neighboring domains have the same color.

1.0.49. (3) Prove that the following identity holds for all positive integer n:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}.$$

Solution→

1.0.50. (3) Prove that the following identity holds for all positive integer n:

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}\cdot y+\ldots+x\cdot y^{n-2}+y^{n-1}$$

1.0.51. (3) Prove that the following identity holds for all positive integer n:

$$1^3+\ldots+n^3=\left(\frac{n\cdot(n+1)}{2}\right)^2.$$

Solution→

1.0.52. (3) Prove that the following identities hold for all positive integer n:

1. $1-\frac{1}{2}+\frac{1}{3}-\ldots-\frac{1}{2n}=\frac{1}{n+1}+\ldots+\frac{1}{2n}$;
2. $\frac{1}{1\cdot 2}+\ldots+\frac{1}{(n-1)\cdot n}=\frac{n-1}{n}$.

1.0.53. (3) Prove that $1\cdot 4+2\cdot 7+3\cdot 10+\cdots+n(3n+1)=n(n+1)^2$.

1.0.54. (5) Express the following sums in closed forms!

1. $1+3+5+7+\ldots+(2n+1)$;
2. $\frac{1}{1\cdot 2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)\cdot(n+2)}$;
3. $1\cdot 2+\ldots+n\cdot(n+1)$;
4. $1\cdot 2\cdot 3+\ldots+n\cdot(n+1)\cdot(n+2)$.

1.0.55. (4) Prove that the following identity holds for all positive integer n:

$$\sqrt{n} \leq 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Hint→

1.0.56. (6) Show that for all positive integer $n \geq 6$ a square can be divided into n squares.

Solution→

1.0.57. (5) $A_1, A_2, \ldots$ are logical statements. What can we say about their truth value if

(a) $A_1 \wedge \forall n \in \mathbb{N}\ A_n \Rightarrow A_{n+1}$?
(b) If $A_1 \wedge \forall n \in \mathbb{N}\ A_n \Rightarrow (A_{n+1} \wedge A_{n+2})$?
(c) If $A_1 \wedge \forall n \in \mathbb{N}\ (A_n \vee A_{n+1}) \Rightarrow A_{n+2}$?
(d) If $\forall n \in N\ \neg A_n \Rightarrow \exists k \in \{1, 2, \ldots, n-1\}\ \neg A_k$?

1.0.58. (4) Prove that

$$1 + \frac{1}{2 \cdot \sqrt{2}} + \ldots + \frac{1}{n \cdot \sqrt{n}} \leq 3 - \frac{2}{\sqrt{n}}.$$

Fibonacci Numbers

1.0.59. (6) Let u_n be the n-th Fibonacci number ($u_0 = 0$, $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $u_5 = 5$, $u_6 = 8$, $\ldots$).

(a) $u_0 + u_2 + \ldots + u_{2n} = ?$
(b) $u_1 + u_3 + \ldots + u_{2n+1} = ?$

1.0.60. (6) Prove that $u_n^2 - u_{n-1}u_{n+1} = \pm 1$.

1.0.61. (3) Let u_n be the n-th Fibonacci number. Prove that

$$\frac{1}{3} \cdot 1,6^n < u_n < 1,7^n.$$

1.0.62. (5) Prove that any two consecutive Fibonacci-numbers are co-prime.

1.0.63. (5) Prove that

$$u_1^2 + \ldots + u_n^2 = u_n u_{n+1}.$$

1.0.64. (6) Express the sums below in closed form!

1. $u_0 + u_3 + \ldots + u_{3n}$;
2. $u_1 u_2 + \ldots + u_{2n-1} u_{2n}$.

1.0.4 Solving Inequalities and Optimization Problems by Inequalities between Means

1.0.65. (6) Let $a, b \geq 0$ and r, s be positive rational numbers with $r + s = 1$. Show that

$$a^r \cdot b^s \leq ra + sb.$$

1.0.66. (3) Prove that if $a, b, c > 0$, then the following inequality holds

$$\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} \geq 3.$$

Solution→

1.0.67. (2) Prove that $\dfrac{x^2}{1+x^4} \leq \dfrac{1}{2}$.

1.0.68. (4) Let $a, b > 0$. For which x is the expression $\dfrac{a + bx^4}{x^2}$ minimal?

Hint→

1.0.69. (3) Let $a_i > 0$. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

1.0.70. (8) Which one is the greater? $1000001^{1000000}$ or $1000000^{1000001}$.

1.0.71. (4) Suppose that the product of three positive numbers is 1.

1. What is the maximum of their sum?
2. What is the minimum of their sum?
3. What is the maximum of the sum of their inverses?
4. What is the minimum of the sum of their inverses?

1.0.72. (4) What is the maximum value of xy if $x, y \geq 0$ and (a) $x + y = 10$; (b) $2x + 3y = 10$?

1.0.73. (2) Prove that $x^2 + \frac{1}{x^2} \geq 2$ if $x \neq 0$.

1.0.74. (4) Which rectangular box has the greatest volume among the ones with given surface area?

Solution→

1.0.75. (4) What is the maximum value of a^3b^2c if a, b, c are non-negative and $a + 2b + 3c = 5$?

1.0.76. (3) Prove that the following inequality holds for all $a, b, c > 0$!

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

1.0.77. (4) Calculate the maximum value of the function $x^2 \cdot (1 - x)$ for $x \in [0, 1]$.

Solution→

1.0.78. (6) Prove that the cylinder with the least surface area among the ones with given volume V is the cylinder whose height equals the diameter of its base.

Solution→

1.0.79. (5) Prove that $n! < \left(\frac{n+1}{2}\right)^n$.

Solution→

1.0.80. (6) What is the maximum of the function $x^3 - x^5$ on the interval $[0, 1]$?

1.0.81. (6) What is the greatest volume of a cylinder inscribed into a right circular cone?

1.0.82. (6) What is the greatest volume of a cylinder inscribed into the unit sphere?

1.0.83. (10) Prove that for any sequence $a_1, a_2, \ldots, a_n$ of positive real numbers,

$$\frac{1}{\frac{1}{a_1}}+\frac{2}{\frac{1}{a_1}+\frac{1}{a_2}}+\frac{3}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}+\ldots+\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}} < 2(a_1+a_2+\ldots+a_n).$$

(KöMaL N. 189., November 1998)

Solution→

1.1 Real Numbers

1.1.1 Field Axioms

1.1.1. (4) Using the field axioms prove the following statements:

If $ab = 0$, then $a = 0$ or $b = 0$;
$-(-a) = a$;
$(a - b) - c = a - (b + c)$;
$-a = (-1) \cdot a$;
$(a/b) \cdot (c/d) = (a \cdot c)/(b \cdot d)$.

1.1.2. (4) Using the field axioms prove the following statements:

$(-a) \cdot b = -(ab)$;
$1/(a/b) = b/a$;
$(a - b) + c = a - (b - c)$.

1.1.3. (4) Using the field axioms prove the following statement: $(-a)(-b) = ab$.

Solution→

1.1.4. (4) Using the field axioms prove the following statements:

1. $(a + b)(c + d) = ac + ad + bc + bd$,
2. $(-x) \cdot y = -\ x \cdot y$.

1.1.5. (5) Prove that if $*$ is an associative binary operation, then any bracketing of the expression $a_1 * a_2 * \ldots * a_n$ has the same value.

1.1.2 Ordering Axioms

1.1.6. (4) Using the field and ordering axioms prove the following statements:

1. If $a < b$, then $-a > -b$;
2. If $a > 0$, then $\frac{1}{a} > 0$;
3. If $a < b$ and $c < 0$, then $ac > bc$.

1.1.7. (3) Prove that for any real numbers a, b we have $|a| - |b| \leq |a - b| \leq |a| + |b|$.

1.1.8. (4) Using the field and ordering axioms prove that $\forall a \in \mathbb{R}\ a^2 \geq 0$.

1.1.9. (5) Show that no ordering can make the field of complex numbers into an ordered field.

Hint→

1.1.10. (4) Define a rational function (a function which can be written as the ratio of two polynomial functions) to be positive if the leading coefficient of its denominator and numerator have the same sign. Prove that this ordering ($r > q \Leftrightarrow r - q$ positive) makes the field of rational functions into an ordered field.

Related problem: 1.1.12

1.1.11. (4) Using the field and ordering axioms prove that $a < b < 0$ implies $\dfrac{1}{b} < \dfrac{1}{a} < 0$.

1.1.3 The Archimedean Axiom

1.1.12. (6) Does the ordered field of rational functions satisfy the Archimedean axiom?

Hint→

Related problem: 1.1.10

1.1.13. (7) Given an ordered field R and a subfield $\mathbb{Q}$ show that if

$$(\forall a, b \in R)\left((1 < a < b < 2) \Rightarrow \left((\exists q \in \mathbb{Q})\ (a < q < b)\right)\right),$$

then R satisfies the Archimedean axiom.

Hint→

1.1.14. (5) In which ordered fields can the floor function be defined?

Answer→

1.1.4 Cantor Axiom

1.1.15. (8) Does the ordered field of rational functions satisfy the Cantor axiom?

Hint→

Related problem: 1.1.10

1.1.16. (5) Answer the following questions. Explain your answer.

1. Can the intersection of a sequence of nested intervals be empty?
2. Can the intersection of a sequence of nested closed intervals be empty?
3. Can the intersection of a sequence of nested closed intervals be a one-point set?
4. Can the intersection of a sequence of nested open intervals be non-empty?
5. Can the intersection of a sequence of nested open intervals be a closed interval?

1.1.17. (8) Using the Cantor axiom give a direct proof of the fact that the subset of irrational numbers is dense in the real line: every open interval contains an irrational number.

1.1.18. (4) Which axioms of the reals are satisfied for the set of rational numbers (with the usual operations and ordering)?

Answer→

1.1.19. (9) Does there exist an ordered field satisfying the Cantor axiom and not satisfying the Archimedean axiom?

1.1.20. (1) Describe the negation of the Archimedean and the Cantor axiom (do not start with negation!).

1.1.21. (2) Describe the intersection of the following sequences of intervals:
1. $I_n = [-\frac{1}{n}, \frac{1}{n}]$, 2. $I_n = (-\frac{1}{n}, \frac{1}{n})$, 3. $I_n = [-5+n, 3+n)$,
4. $I_n = [2-\frac{1}{n}, 3+\frac{1}{n}]$, 5. $I_n = (2-\frac{1}{n}, 3+\frac{1}{n})$, 6. $I_n = [2-\frac{1}{n}, 3+\frac{1}{n})$,
7. $I_n = [0, \frac{1}{n}]$, 8. $I_n = (0, \frac{1}{n})$, 9. $I_n = [0, \frac{1}{n})$, 10. $I_n = (0, \frac{1}{n}]$.

1.1.5 The Real Line, Intervals

1.1.22. (3) Prove that $\sqrt{2}$ is irrational.

1.1.23. (4) Prove that
1. $\sqrt{3}$ is irrational; 2. $\frac{\sqrt{2}}{\sqrt{3}}$ is irrational; 3. $\frac{\frac{\sqrt{2}+1}{2}+3}{4}+5$ is irrational!

1.1.24. (3) Let $a, b \in \mathbb{Q}$ and c, d be irrational. What can we say about the rationality of $a+b$, $a+c$, $c+d$, ab, ac and cd?

1.1.25. (3) Prove that there is a rational and an irrational number in every open interval.

1.1.26. (2) How many (a) maxima (b) upper bounds of a set of real numbers can have?

1.1.27. (2) Determine the minimum, maximum, infimum, supremum of the following sets (if they have any)!
1. $[1,2]$, 2. $(1,2)$, 3. $\{\frac{1}{n} : n \in \mathbb{N}^+\}$, 4. $\mathbb{Q}$, 5. $\{\frac{1}{n} + \frac{1}{\sqrt{n}} : n \in \mathbb{N}^+\}$,
6. $\{\sqrt[n]{2} : n \in \mathbb{N}^+\}$, 7. $\{x : x \in (0,1) \cap \mathbb{Q}\}$, 8. $\{\frac{1}{n} + \frac{1}{k} : n, k \in \mathbb{N}^+\}$,
9. $\{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N}^+\}$, 10. $\{n + \frac{1}{n} : n \in \mathbb{N}^+\}$

1.1.28. (2) Are the following sets bounded from above or from below? What is the maximum, minmimum, supremum and infimum? Which set is convex?

$$\emptyset \qquad \{1,2,3,\dots\} \qquad \{1, -1/2, 1/3, -1/4, 1/5, \dots\} \qquad \mathbb{Q} \qquad \mathbb{R}$$

$$[1,2) \qquad (2,3] \qquad [1,2) \cup (2,3]$$

1.1.29. (2) Let H be a subset of the reals. Which properties of H are expressed by the following formulas?

1. $(\forall x \in \mathbb{R})(\exists y \in H)(x < y)$;
2. $(\forall x \in H)(\exists y \in \mathbb{R})(x < y)$;
3. $(\forall x \in H)(\exists y \in H)(x < y)$.

1.1.30. (3) Let $A \cap B \neq \emptyset$. What can we say about the connections among $\sup A$, $\sup B$ and $\sup(A \cup B)$, $\sup(A \cap B)$ and $\sup(A \setminus B)$?

1.1.31. (3) Which subsets $H \subset \mathbb{R}$ satisfy that
(a) $\inf H < \sup H$; (b) $\inf H = \sup H$; (c) $\inf H > \sup H$?

1.1.32. (5) What are the suprema and infima of the following sets?
a) $\{\frac{1}{n} | n \in \mathbb{N}\}$.
b) $\{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$.
c) $\{\frac{1}{n} | n \in \mathbb{N}\} \cup \{\frac{-1}{n} | n \in \mathbb{N}\}$.
d) $\{\frac{1}{n^n} | n \in \mathbb{N}\} \cup \{2, 3\}$.
e) $\{\frac{\cos n}{n^n} | n \in \mathbb{N}\} \cup [-6, -5] \cup (100, 101)$.

1.1.33. (5) Let H, K be non-empty subsets of the real line $\mathbb{R}$. What is the logical connection between the following two statements?
a) $\sup H < \inf K$;
b) $\forall x \in H \ \exists y \in K \ x < y$.

1.1.34. (4) Let $a_n = \sqrt{n+1} + (-1)^n \sqrt{n}$.

$$\inf\{a_n | n \in \mathbb{N}\} = ?$$

1.1.35. (5) Let A, B be subsets of the real line $\mathbb{R}$ such that $A \cup B = (0, 1)$. Does it imply that

$$\inf A = 0 \qquad \text{or} \qquad \inf B = 0 \qquad ?$$

1.1.36. (7) Prove that all convex subset of $\mathbb{R}$ are intervals.

1.1.6 Completeness Theorem, Connectivity, Topology of the Real Line

1.1.37. (7) Does the ordered field of the rational functions satisfy the completeness theorem: all non-empty set has a supremum?

Hint→ Solution→

Related problem: 1.1.10

1.1.38. (6) Prove that if an ordered field satisfies the completeness theorem, then the Archimedean axiom holds.

Hint→

1.1.39. (6) Prove that if an ordered field satisfies the completeness theorem, then the Cantor axiom holds.

Hint→

1.1.40. (9) Define recursively the sequence $x_{n+1} = x_n \left(x_n + \frac{1}{n}\right)$ for any x_1. Show that there is exactly one x_1 for which $0 < x_n < x_{n+1} < 1$ for any n.
(IMO 1985/6)

Hint→

1.1.7 Powers

1.1.41. (6) Prove that $(a^x)^y = a^{xy}$ if $a > 0$ and $x, y \in \mathbb{Q}$.

1.1.42. (6) Prove that $(1+x)^r \leq 1 + rx$ if $r \in \mathbb{Q}$, $0 < r < 1$ and $x \geq -1$.

Solution→

1.1.43. (6) Can x^y be (ir)rational if x is (ir)rational and y is (ir)rational (these are 8 exercises)?

Chapter 2

Convergence of Sequences

2.1 Theoretical Exercises

2.1.1. (3) Suppose $0 < a_n \to 0$. Prove that there are infinitely many n for which $a_n > a_{n+r}$ for all $r = 1, 2, \ldots$.

2.1.2. (2) $0 < a_n < 1$ for all $n \in \mathbb{N}$. Does it imply that $a_n^n \to 0$?

2.1.3. (2) Suppose that $a_{2n} \to B$, $a_{2n+1} \to B$. Does it imply that $a_n \to B$?

2.1.4. (3) Does

$$\frac{a_n}{3 - a_n} \to 2$$

imply $a_n \to 2$?

2.1.5. (3) Prove that $x_n \to a \neq 0$ implies $\lim \frac{x_{n+1}}{x_n} = 1$.

2.1.6. (4) Prove that if $y_n \to 0$ and $Y = \lim \frac{y_{n+1}}{y_n}$ exist, then $y \in [-1, 1]$.

2.1.7. (2) Let a_n be a sequence of real numbers. Write down the negation of the statement $\lim a_n = 7$ (do not start with negation!).

2.1.8. (4) Show that the sequence a_n is bounded if and only if for all sequences $b_n \to 0$ the sequence $a_n b_n$ also tends to 0.

2.1.9. (4) Give an example of a sequence $a_n \to \infty$ such that $\forall k = 1, 2, \ldots$ $(a_{n+k} - a_n) \to 0$.

2.1.10. (4) Give examples of sequences a_n, with the property $\frac{a_{n+1}}{a_n} \to 1$ such that

1. a_n is convergent; 2. $a_n \to \infty$;
3. $a_n \to -\infty$; 4. a_n is oscillating.

2.1.11. (5) Suppose that $a_n b_n \to 1$, $a_n + b_n \to 2$. Does it imply that $a_n \to 1$, $b_n \to 1$?

2.1.12. (4) Show that every convergent sequence has a minimum or a maximum.

Hint→

2.1.13. (3) Prove that $a_n \ge 0$ and $a_n \to a$ implies $\sqrt{a_n} \to \sqrt{a}$.

2.1.14. (3) Show that every sequence tending to infinity has a minimum.

2.1.15. (3) Show that every sequence tending to minus infinity has a maximum.

Related problem: 2.1.12

2.1.16. (2) Prove that $a_n \to \infty$ implies that $\sqrt{a_n} \to \infty$.

2.1.17. (3) Suppose that $a_n \to -\infty$, and let $b_n = \max\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Show that $b_n \to -\infty$.

2.1.18. (2) Is it true that if x_n is convergent, y_n is divergent, then $x_n y_n$ is divergent?

Solution→

2.1.19. (3) Let a_n be a sequence and a be a number. What are the implications among the following statements?

a) $\forall \varepsilon > 0\ \exists N\ \forall n \ge N\ |a_n - a| < \varepsilon$.
b) $\forall \varepsilon > 0\ \exists N\ \forall n \ge N\ |a_n - a| \ge \varepsilon$.
c) $\exists \varepsilon > 0\ \forall N\ \forall n \ge N\ |a_n - a| < \varepsilon$.
d) $\forall \varepsilon > 0\ \forall N\ \forall n \ge N\ |a_n - a| < \varepsilon$.
e) $\exists \varepsilon' > 0\ \forall 0 < \varepsilon < \varepsilon'\ \exists N\ \forall n \ge N\ |a_n - a| < \varepsilon$.

2.1.20. (3)

a) $a_n \to 1$. Does it imply that $a_n^n \to 1$?

b) $a_n > 0, a_n \to 0$. Does it imply that $\sqrt[n]{a_n} \to 0$?

c) $a_n > 0, a_n \to a > 0$. Does it imply that $\sqrt[n]{a_n} \to 1$?

d) $c_n d_n \to 0$. Does it imply that $c_n \to 0$ or $d_n \to 0$?

2.1.21. (1) Show that 1. $a_n \to a \iff (a_n - a) \to 0$, 2. $a_n \to 0 \iff |a_n| \to 0$.

2.1.22. (1) Show that $\lim_{n\to\infty} a_n = \infty \iff \forall K \in \mathbb{R}$ only finitely many members of (a_n) are smaller than K.

2.1.23. (2) Show that if $\forall n \geq n_0\ a_n \leq b_n$ and $a_n \to \infty$, then $b_n \to \infty$.

2.1.24. (4) Give examples showing that if $a_n \to 0$ and $b_n \to +\infty$, then $a_n b_n$ is critical.

2.1.25. (1) Show that if $a_n \to 0$ and $a_n \neq 0$, then $\frac{1}{|a_n|} \to \infty$.

2.1.26. (3) Which of the following statements is equivalent to the negation of $a_n \to A$? What is the meaning of the rest? What are the implications among them?

1. For all $\varepsilon > 0$ there are infinitely many members of a_n outside of $(A - \varepsilon, A + \varepsilon)$.
2. There is an $\varepsilon > 0$ such that there are infinitely many members of a_n outside of $(A - \varepsilon, A + \varepsilon)$.
3. For all $\varepsilon > 0$ there are only finitely many members of a_n in the interval $(A - \varepsilon, A + \varepsilon)$.
4. There is an $\varepsilon > 0$ such that there are only finitely many members of a_n in the interval $(A - \varepsilon, A + \varepsilon)$.

2.1.27. (3) Is there a sequence of irrational numbers converging to (a) 1, (b) $\sqrt{2}$?

Solution→

2.1.28. (3) Give examples such that $a_n - b_n \to 0$ but $a_n/b_n \not\to 1$, and $a_n/b_n \to 1$ but $a_n - b_n \not\to 0$.

2.1.29. (2) Prove that if (a_n) is convergent, then $(|a_n|)$ is convergent, too. Does the reverse implication also hold?

2.1.30. (3) Does $a_n^2 \to a^2$ imply that $a_n \to a$? And does $a_n^3 \to a^3$ imply that $a_n \to a$?

Solution→

2.1.31. (4) Consider the sequence s_n of arithmetic means

$$s_n = \frac{a_1 + \ldots + a_n}{n}$$

corresponding to the sequence a_n. Show that if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} s_n = a$. Give an example when (s_n) is convergent, but (a_n) is divergent.

2.1.32. (5) Prove that if $a_n \to \infty$, then $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \to \infty$.

Related problem: 2.1.31

2.1.33. (5) Prove that if $\forall n\ a_n > 0$ and $a_n \to b$, then $\sqrt[n]{a_1 a_2 \ldots a_n} \to b$.

Related problem: 2.1.31

2.1.34. (4) Consider the definition of $a_n \to b$:

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n \ge n_0)(|a_n - b| < \varepsilon).$$

Changing the quantifiers and their order we can produce the following statements:

1. $(\forall \varepsilon > 0)(\exists n_0)(\exists n \ge n_0)(|a_n - b| < \varepsilon)$;
2. $(\forall \varepsilon > 0)(\forall n_0)(\forall n \ge n_0)(|a_n - b| < \varepsilon)$;
3. $(\exists \varepsilon > 0)(\exists n_0)(\exists n \ge n_0)(|a_n - b| < \varepsilon)$;
4. $(\exists n_0)(\forall \varepsilon > 0)(\forall n \ge n_0)(|a_n - b| < \varepsilon)$;
5. $(\forall n_0)(\exists \varepsilon > 0)(\exists n \ge n_0)(|a_n - b| < \varepsilon)$.

Which properties of the sequence (a_n) are expressed by these statements? Give examples of sequences (if they exist) satisfying these properties.

2.1.35. (4) Consider the definition of $a_n \to \infty$:

$$(\forall P)(\exists n_0)(\forall n \ge n_0)(a_n > P).$$

Changing the quantifiers and the orders we can produce the following statements:

1. $(\forall P)(\exists n_0)(\exists n \geq n_0)(a_n > P)$;
2. $(\forall P)(\forall n_0)(\forall n \geq n_0)(a_n > P)$;
3. $(\exists P)(\exists n_0)(\forall n \geq n_0)(a_n > P)$;
4. $(\exists P)(\exists n_0)(\exists n \geq n_0)(a_n > P)$;
5. $(\exists n_0)(\forall P)(\forall n \geq n_0)(a_n > P)$;
6. $(\forall n_0)(\exists P)(\exists n \geq n_0)(a_n > P)$.

Which properties of the sequence (a_n) are expressed by these statements? Give examples of sequences (if they exist) satisfying these properties.

2.1.36. (4) Construct sequences (a_n) with all possible limit behavior (convergent, tending to infinity, tending to minus infinity, oscillating), while $a_{n+1} - a_n \to 0$ holds.

2.1.37. (3) Prove that if $a_n \to \infty$ and (b_n) is bounded, then $(a_n + b_n) \to \infty$.

2.1.38. (3) Prove that if (a_n) has no subsequence tending to infinity, then (a_n) is bounded from above.

2.1.39. (4) Prove that if (a_{2n}), (a_{2n+1}), (a_{3n}) are convergent, then a_n is convergent, too.

2.1.40. (3) Prove that if $a_n \to a > 1$, then $(a_n^n) \to \infty$.

2.1.41. (4) Prove that if $a_n \to a$, with $|a| < 1$, then $(a_n^n) \to 0$.

2.1.42. (4) Prove that if $a_n \to a > 0$, then $\sqrt[n]{a_n} \to 1$.

2.1.43. (3) Prove that if $(a_n + b_n)$ is convergent and (b_n) is divergent, then (a_n) is also divergent.

Hint→

2.1.44. (3) Is it true that if $(a_n \cdot b_n)$ is convergent and (b_n) is divergent, then (a_n) is divergent?

2.1.45. (3) Is it true that if (a_n / b_n) is convergent and (b_n) is divergent, then (a_n) is divergent?

2.1.46. (3) Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Prove that $\max(a_n, b_n) \to \max(a, b)$.

2.1.47. (4) Let $a_k \neq 0$ and $p(x) = a_0 + a_1 x + \ldots + a_k x^k$. Prove that

$$\lim_{n\to+\infty} \frac{p(n+1)}{p(n)} = 1.$$

Solution→

2.1.48. (4) Show that if $a_n > 0$ and $a_{n+1}/a_n \to q$, then $\sqrt[n]{a_n} \to q$.

2.1.49. (4) Give an example of a positive sequence (a_n) for which $\sqrt[n]{a_n} \to 1$, but a_{n+1}/a_n does not tend to 1.

2.1.50. (5) There are 8 possibilities for a sequence, according to monotonicity, boundedness and convergence. Which of these 8 classes are non-empty?

2.1.51. (5) Assume that $a_n \to a$ and $a < a_n$ for all n. Prove that a_n can be rearranged to a monotone decreasing sequence.

Hint→

2.1.52. (6) The sequence (a_n) satisfies the inequality $a_n \le (a_{n-1} + a_{n+1})/2$ for all $n > 1$. Prove that (a_n) cannot be oscillating.

2.1.53. (6) Prove that if (a_n) is convergent and $(a_{n+1} - a_n)$ is monotone, then $n \cdot (a_{n+1} - a_n) \to 0$. Give an example for a convergent sequence (a_n) for which $n \cdot (a_{n+1} - a_n)$ does not tend to 0.

2.1.54. (4) Prove that if the sequence (a_n) has no convergent subsequence, then $|a_n| \to \infty$.

Solution→

2.1.55. (5) Prove that if the sequence (a_n) is bounded and all of its convergent subsequences tend to b, then $a_n \to b$.

2.1.56. (4) Prove that if all subsequence of a sequence (a_n) have a subsequence tending to b, then $a_n \to b$.

2.1.57. (4) Does $a_{n+1} - a_n \to 0$ imply that $a_{2n} - a_n \to 0$?

2.1.58. (4) Give examples such that $a_n \to \infty$ and
1. $a_{2n} - a_n \to 0$; 2. $a_{n^2} - a_n \to 0$; 3. $a_{2^n} - a_n \to 0$.

2.1.59. (5) Prove that every sequence can be obtained as the product of a sequence tending to 0, and a sequence tending to infinity.

2.1.60. (5) Assume that $a_n \to 1$. What can we say about the limit of the sequence (a_n^n)?

2.1.61. (5) How would you define 0^0, ∞^0 and 1^∞? Explain it.

2.2 Order of Sequences, Threshold Index

2.2.1. (3) Prove that

$$1 \cdot \frac{1}{2^2} \cdots \frac{1}{3^3} \cdot \ldots \cdot \frac{1}{n^n} < \left(\frac{2}{n+1}\right)^{\frac{n(n+1)}{2}}.$$

2.2.2. (5) Prove that $n^{n+1} > (n+1)^n$ if $n > 2$.

Solution→

Related problem: 2.6.8

2.2.3. (8) Prove that

$$\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \ldots \cdot \sqrt[2^n]{2^n} < n+1.$$

Solution→

2.2.4. (5) Prove that $2^n > n^k$ holds for all sufficiently (depending on k) large n.

Solution→

2.2.5. (5) Prove that the following two statement are true for n big enough.
1. $2^n > n^3$, 2. $n^2 - 6n - 100 > 8n + 11$

2.2.6. (5) Find an $n_o \in N$ such that $\forall n > n_o$ the following statements hold:
1. $n^2 - 15n + 124 > 14512n$, 2. $n^3 - 16n^2 + 25 > 15n + 32162$,
3. $(1.01)^n > 1000$, 4. $n! > n^5$.

2.2.7. (5) Find an $n_o \in N$ such that $\forall n > n_o$ the following holds:
1. $(1.01)^n > n$, 2. $(1.01)^n > n^2$, 3. $(1.0001)^n > 1000 \cdot \sqrt{n}$,
4. $100^n < n!$ 5. $\frac{1}{2} < \frac{2n^2+3n-2}{3n^2-4n+20} < 1$, 6. $3^n - 1000 \cdot 2^n > n^3 + 100n^2$,
7. $\sqrt{n+1} - \sqrt{n} > \frac{1}{n}$, 8. $n! > \left(\frac{n}{2}\right)^{\frac{n}{2}}$, 9. $n\left(\frac{n}{e}\right)^n > n! > \left(\frac{n}{e}\right)^n$.

2.2.8. (4) Find an $n_o \in N$ such that $\forall n > n_0$ the following holds:
1. $\sqrt{n+1} - \sqrt{n} < 0.1$ 2. $\sqrt{n+3} - \sqrt{n} < 0.01$
3. $\sqrt{n+5} - \sqrt{n+1} < 0.01$ 4. $\sqrt{n^2+5} - n < 0.01$.

2.2.9. (4) Prove that the sequence $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{a_n}$ has a member which is greater than 100.

2.2.10. (4) Prove that for the sequence $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{a_n}$ we have $a_{10001} > 100$ (see the 2.2.9 exercise and its solution.)

Solution→

Related problems: 2.2.9, 2.5.19

2.2.11. (5) Determine the limit of the following recursively defined sequence!
$a_1 = 0,\ a_{n+1} = 1/(1 + a_n)\ (n = 1, 2, \ldots)$.

Hint→

2.2.12. (2) Using the definition calculate the limit (if exists) of the following sequences. Give a threshold index to $\varepsilon = 10^{-4}$!

$$1/\sqrt{n}; \qquad (-1)^n$$

2.2.13. (4) Using the definition calculate the limit (if exists) of the following sequences. Give a threshold index to $\varepsilon = 10^{-6}$!

$$\frac{2n+1}{n+1}; \qquad \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$$

2.2.14. (4) Using the definition calculate the limit (if exists) of the following sequences. Give a threshold index to $\varepsilon = 10^{-4}$, to $P = 10^6$ and to $P = -10^6$.

$$\frac{1+2+\ldots+n}{n^2}; \qquad n^2 - n^3; \qquad n\left(\sqrt{n+1} - \sqrt{n}\right); \qquad \sin n$$

2.2.15. (4) Find an $n_0 \in N$ such that $\forall n > n_0$ the following holds:
1. $n^2 > 6n + 15$ 2. $n^2 > 6n - 15$ 3. $n^3 > 6n^2 + 15n + 37$
4. $n^3 > 6n^2 - 15n + 37$ 5. $n^3 - 4n + 2 > 6n^2 - 15n + 37$
6. $n^5 - 4n^2 + 2 > 6n^3 - 15n + 37$
7. $n^5 + 4n^2 - 2 > 6n^3 + 15n - 37$.

2.2.16. (4) Find an $n_0 \in N$ such that $\forall n > n_0$ the following holds:
1. $2^n > n^4$, 2. $(1 + \frac{1}{n})^n \geq 2$; 3. $1,01^n > 100$, 4. $1,01^n > 1000$;
5. $0,9^n < \frac{1}{100}$; 6. $\sqrt[n]{2} < 1,01$, 7. $\sqrt{n+1} - \sqrt{n} < \frac{1}{100}$,
8. $\sqrt{n^2+5} - n < 0,01$, 9. $n^7 > 100n^5$,
10. $n^8 + n^3 - 10n^2 > n^5 + 1000n$.

2.2.17. (4) Calculate the limit of the following sequences and find an n_0 threshold for $\varepsilon > 0$.

1. $1/\sqrt{n}$; 2. $(2n+1)/(n+1)$; 3. $(5n-1)/(7n+2)$; 4. $1/(n-\sqrt{n})$;

5. $(1 + \ldots + n)/n^2$; 6. $(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n})/n^{4/3}$;

7. $n \cdot \left(\sqrt{1 + (1/n)} - 1\right)$; 8. $\sqrt{n^2+1} + \sqrt{n^2-1} - 2n$;

9. $\sqrt[3]{n+2} - \sqrt[3]{n-2}$; 10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \ldots + \dfrac{1}{(n-1) \cdot n}$.

2.2.18. (4) Find an n_0 threshold for P for the following sequences.
1. $n - \sqrt{n}$; 2. $(1 + \ldots + n)/n$; 3. $(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n})/n$;
4. $\dfrac{n^2 - 10n}{10n + 100}$; 5. $2^n/n$;

2.2.19. (5) Prove that there is an N natural number such that $\forall n > N$ the following inequality holds:

$$\left(\frac{3}{2}\right)^n > n^2.$$

2.2.20. (5) Find an N natural number such that $\forall n > N$ the following inequality holds:

a) $10^n + 11^n + 12^n < 13^n$; *b)* $1.01^n > n$; *c)* $\sqrt{n} + \sqrt{n+2} + \sqrt{n+4} < n^{0,51}$.

2.2.21. (4) Find an N natural number such that $\forall n > N$ the following inequality holds: $1.0001^n > n^{100}$.

2.2.22. (4) Find an N natural number such that $\forall n > N$ the following inequality holds:

$$\frac{1}{n-5\sqrt{n}} > \frac{10n^2}{2^n - 100}.$$

2.3 Limit Points, liminf, limsup

2.3.1. (3) Find a non-convergent sequence with exactly one limit point.

Solution→

2.3.2. (1) Given $a_1, \dots, a_p \in \mathbb{R}$, find a sequence with exactly these limit points.

2.3.3. (2) Calculate the limit points of the sets $B(0,1)$, $\dot{B}(0,1)$, $\mathbb{N}$, $\mathbb{Q}$ and $\{1/n : n \in \mathbb{N}\}$!

2.3.4. (5) Prove that the set of limit points of a sequence (or a set) is closed.

2.3.5. (6) Find a sequence such that the set of limit points of it is $[0,1]$.

Solution→

2.3.6. (6) Prove that a limit point of the set of limit points of a set is a limit point of the original set.

2.3.7. (2) What are the limit points, limsup and liminf of the following sequences?

$$\sqrt[n]{n}; \qquad (-1)^n + \frac{1}{n}; \qquad \{\sqrt{n}\}$$

2.3.8. (2) What is the limsup and liminf of the following sequence?

$$a_n = \frac{n^k}{2^n}.$$

2.3.9. (4) Using the definition of lim sup and lim inf prove that $\liminf a_n \leq \limsup a_n$.

2.3.10. (4) Prove that if (a_n) is convergent and (b_n) is an arbitrary sequence, then

$$\overline{\lim}(a_n + b_n) = \lim a_n + \overline{\lim}\, b_n.$$

2.3.11. (3) Prove that if $a_n \to a > 0$ and (b_n) is an arbitrary sequence, then

$$\underline{\lim}(a_n \cdot b_n) = a \cdot \underline{\lim}\, b_n \quad \text{and}$$

$$\overline{\lim}(a_n \cdot b_n) = a \cdot \overline{\lim}\, b_n.$$

2.3.12. (5) Prove that if

(i) $a_n \to a \geq 1$ and (b_n) is bounded, then

$$\overline{\lim}\, a_n^{b_n} = a^{\overline{\lim}\, b_n} \quad \text{and} \quad \underline{\lim}\, a_n^{b_n} = a^{\underline{\lim}\, b_n}.$$

(ii) $a_n \to a \leq 1$ and (b_n) is bounded, then

$$\overline{\lim}\, a_n^{b_n} = a^{\underline{\lim}\, b_n} \quad \text{and} \quad \underline{\lim}\, a_n^{b_n} = a^{\overline{\lim}\, b_n}.$$

2.3.13. (4) Prove that if the sequence (a_n) is bounded with $\liminf a_n > 0$ and $b_n \to 0$, then $a_n^{b_n} \to 1$.

2.3.14. (5) Prove that for an arbitrary sequence of real numbers $a_1, a_2, \ldots$

$$\liminf \frac{a_1 + a_2 + \ldots + a_n}{n} \geq \liminf a_n$$

and

$$\limsup \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \limsup a_n.$$

2.3.15. (5) Prove that if $a_n \to a$, then

$$\inf \big\{ \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} : n \in \mathbb{N} \big\} = a.$$

2.4 Calculating the Limit of Sequences

2.4.1. (1) Guess the limits, and prove using the definition:

1. $\lim \frac{(-1)^n}{n} = ?$ 2. $\lim \frac{1}{n!} = ?$
3. $\lim \frac{2n}{n^2+1} = ?$ 4. $\lim b^n = ?$ for $0 < b < 1$.

2.4.2. (2) Guess the limit, and prove using the definition:

$$\lim \frac{n}{2^n} = ?$$

2.4.3. (2) Determine the limit of $\dfrac{n^2+1}{n+1} - an$ for all values of a.

2.4.4. (3) Determine the limit of $\sqrt{n^2 - n + 1} - an$ for all values of a.

2.4.5. (3) Prove that $\sqrt[n]{2} \to 1$.

2.4.6. (4) Calculate $\lim_{n\to\infty} \sqrt[n]{2^n - n}$.

Solution→

2.4.7. (4) Guess the limits, and prove using the definition:

$$\lim \frac{2^n}{n!} = ?$$

2.4.8. (3)

$$\lim \frac{n^2 + 6n^3 - 2n + 10}{-4n - 9n^3 + 10^{10}} = ?$$

2.4.9. (3)

$$\lim \frac{n + 7\sqrt{n}}{2n\sqrt{n} + 3} = ?$$

2.4.10. (4) Calculate the following:

$$\lim \frac{n^{100}}{1,1^n} = ?$$

Hint→

2.4.11. (5) Calculate the limit of the sequence $\sqrt[n]{n}$.

2.4.12. (4) Calculate the limit of the sequence $\sqrt[n]{n!}$.

2.4.13. (4) Calculate the limit of the following sequences.

1. $\dfrac{n^5-n^3+1}{3n^5-2n^4+8}$; 2. $\sqrt{n^4+n^2}-n^2$; 3. $\sqrt[n]{6^n-5^n}$.

2.4.14. (4) Calculate the limit of the following sequences.

1. $\sqrt[n]{3}$ 2. $\sqrt[n]{\frac{1}{n}}$ 3. $\left(\dfrac{1+\log 2}{n}\right)^n$ 4. $\sqrt[n]{2^n+n}$
5. $\sqrt[n]{1+2+3+\ldots+n}$ 6. $\sqrt[n]{1^n+2^n+3^n+\ldots+100^n}$
7. $\dfrac{n^2+(n+2)^3}{n^2-\sqrt{(n^2+1)(n^4+2)}}$ 8. $\dfrac{n^{100}2^n+3^n}{\left(\sqrt{4^n+1}-2^n+n^5\right)(5^{n+6}-8)}$

2.4.15. (4) Calculate the limit of the following sequences.

1. $\dfrac{3n+16}{4n-25}$, 2. $n\cdot\left(\sqrt{1+\dfrac{1}{n}}-1\right)$, 3. $\dfrac{1}{n}\cdot\dfrac{n^2+1}{n^3+1}$, 4. $\dfrac{5-2n^2}{4+n}$,

5. $\dfrac{\sin(n)+n}{n}$, 6. $\dfrac{2n^3+3\sqrt{n}}{1-n^3}$, 7. $\sqrt[n]{n+5^n}$, 8. $\dfrac{2^n+n!}{n^n-n^{1000}}$,

9. $\sqrt[n]{n^n-5^n}$, 10. $\dfrac{\sin(n)}{n}$, 11. $\dfrac{5n^2+(-1)^n}{8n}$, 12. $\dfrac{6n+2n^2\cdot(-1)^n}{n^2}$.

2.4.16. (4) Calculate the limit of the following sequences.

1. $\sqrt[n]{2n+\sqrt{n}}$, 2. $\dfrac{n^7-6n^6+5n^5-n-1}{n^3+n^2+n+1}$, 3. $\dfrac{n^3+n^2\sqrt{n}-\sqrt{n}+1}{2n^3-6n+\sqrt{n}-2}$,
4. $\sqrt[n]{\dfrac{1}{n}-\dfrac{2}{n^2}}$,

5. $\sqrt[n]{2^n+3^n}$, 6. $\dfrac{\sqrt{2n+1}}{\sqrt{3n+4}}$, 7. $\log\dfrac{n+1}{n+2}$, 8. $\dfrac{7^n-7^{-n}}{7^n+7^{-n}}$,

9. $\dfrac{(2n+3)^5\cdot(18n+17)^{15}}{(6n+5)^{20}}$, 10. $\dfrac{\sqrt{4n^2+2n+100}}{\sqrt[3]{6n^3-7n^2+2}}$, 11. $\dfrac{\sqrt[4]{n^3+6}}{\sqrt[3]{n^2+3n-2}}$,

12. $n\cdot(\sqrt{n+1}-\sqrt{n})$, 13. $\dfrac{2^n+5^n}{3^n+1}$, 14. $n\cdot(\sqrt{n^2+n}-\sqrt{n^2-n})$.

2.4.17. (4)

$$\lim \frac{1}{n(\sqrt{n^2-1}-n)} = ?$$

Solution→

2.4.18. (4)

$$\lim \left(\frac{4n+1}{4n+8}\right)^{3n+2} = ?$$

2.4.19. (4) Let $a > 0$.

$$\lim \sqrt[n]{n+a^n} = ?$$

Hint→

2.4.20. (7) Is the sequence

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}$$

convergent?

2.4.21. (4)

$$\lim \frac{1-2+3-4+\ldots-2n}{2n+1} = ?$$

2.4.22. (5) Is

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}$$

convergent?

Hint→

2.4.23. (4) Calculate the following:

$$\lim \left(\sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdot \ldots \cdot \sqrt[2^n]{2}\right) = ?$$

1.4-8c

2.4.24. (4) Is

$$\sqrt[n]{n^2+\cos n}$$

convergent?

Solution→

2.4.25. (4) Calculate the following

$$\lim \sqrt[n]{2^n + \sin n}.$$

2.4.26. (5) Calculate the following

$$\lim \frac{\sqrt[n]{n!}}{n}.$$

2.4.27. (4) Calculate the limit of the following sequences.

1. $\dfrac{6n^4 + 2n^2 \cdot (-1)^n}{n^4}$, 2. $\sqrt{n^2+2} + \sqrt{n^2-2} - 2n$;

3. $\dfrac{\sqrt[n]{n^n - 5^n}}{n}$, 4. $n \cdot (\sqrt{n^2+n} - \sqrt{n^2-n})$.

2.4.28. (5) Suppose that $a_1, a_2, \ldots, a_k > 0$. Calculate the limit of the sequence $\sqrt[n]{a_1^n + a_2^n + \ldots + a_k^n}$.

2.4.29. (5) Calculate the limit of the sequence $\left(\sqrt{n + \sqrt{n + \sqrt{n}}} - \sqrt{n}\right)$.

2.4.30. (4) Let $|a|, |b| < 1$.

$$\lim \frac{1 + a + a^2 + \ldots + a^n}{1 + b + b^2 + \ldots + b^n} = ?$$

2.4.31. (4) Calculate:

1. $\lim \sqrt[n^2]{1^2 + 2^n + 3^n + \ldots + n^n} = ?$
2. $\lim \sqrt[n]{1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}} = ?$
3. $\lim \dfrac{\dfrac{1}{n^2} + \dfrac{1}{(n+2)^3}}{\dfrac{1}{n!} - \dfrac{1}{\sqrt{(n^2+1)(n^4+2)}}} = ?$

2.4.32. (4)

$$\lim \frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}}{\sqrt{n}} = ?$$

2.4.33. (4)

$$\lim_{n\to\infty} \sqrt[n]{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}} = ?$$

2.5 Recursively Defined Sequences

2.5.1. (2) Let $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{1+\frac{1}{a_n}}$. Prove that a_n is monotone increasing.

2.5.2. (3) Study the sequence $a_1 = 0.9 \quad a_{n+1} = a_n - a_n^2$. Is it monotone? bounded? Does it have a limit?

2.5.3. (4) Let $a_1 = 0.9$, $a_{n+1} = a_n - a_n^5$. Is there a member of the sequence which is smaller than $\frac{1}{10^{10}}$?

2.5.4. (4) Define the sequence $(a_n)_{n=1}^{\infty}$by the recursion

$$a_1 = 10; \qquad a_{n+1} = \frac{2a_n}{a_n+1}.$$

(a) Prove that the sequence is bounded by giving explicit upper and lower bounds.

(b) Prove that $a_n \to 1$. Check the definition and find n_0 for all $\varepsilon > 0$.

2.5.5. (3) Let

$$x_1 = 1, \qquad x_{n+1} = \sqrt{3x_n}.$$

Is x_n convergent? If yes, what is the limit?

2.5.6. (3) Study the sequence $a_1 = 0, \quad a_{n+1} = \sqrt{2+a_n}$. Is it monotone? bounded? Does it have a limit?

2.5.7. (3) Determine the limit of the following recursively defined sequences!

1. $a_1 = 0,\ a_{n+1} = 1/(2 - a_n)\ (n = 1, 2, \ldots);$
2. $a_1 = 0,\ a_{n+1} = 1/(4 - a_n)\ (n = 1, 2, \ldots);$
3. $a_1 = 0,\ a_{n+1} = 1/(1 + a_n)\ (n = 1, 2, \ldots);$
4. $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2}\,\sqrt{a_n}\ (n = 1, 2, \ldots);$

2.5.8. (6) Let $A > 0$, $x_1 = 1$ and $x_{n+1} = \dfrac{x_n + \frac{A}{x_n}}{2}$. Prove that $x_n \to \sqrt{A}$.

2.5.9. (3) Let $x_1 = 1$, $x_{n+1} = \sqrt{x_n + 2}$. Prove that
(a) The sequence x_n is monotone increasing;
(b) The sequence x_n is bounded;
(c) The limit of the sequence x_n is 2.

2.5.10. (4) Let $x_1 = 1$, $x_{n+1} = \dfrac{6}{5 - x_n}$. Calculate the limit of x_n.

2.5.11. (4) Let $a_0 = 0, a_1 = 1$, and $a_{n+2} = \frac{a_n + a_{n+1}}{2}$ $(n = 0, 1, 2, \ldots)$. $\lim a_n =?$

2.5.12. (2) Let $a_1 = 100$, $a_{n+1} = \sqrt{a_n + 6}$. Prove that
(a) the sequence a_n is monotone;
(b) the sequence a_n is bounded.
(c) What is the limit of the sequence a_n?

2.5.13. (4) Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n^{100}}$ if $n \geq 1$. Is this sequence bounded? If yes what is the limit?

2.5.14. (4) Define the sequence $(x_n)_{n=1}^{\infty}$ by the following recursion: let $x_1 = 3\sqrt{2}$, and $x_{n+1} = \dfrac{8}{6 - x_n}$ if $n \geq 1$. What is the limsup of the sequence?

2.5.15. (5) Let $a_1 = 10$ and $a_{n+1} = \dfrac{2a_n}{a_n^2 + 1}$. $\lim a_n =?$

2.5.16. (5) Does the sequence

$$a_1 = 1, \qquad a_{n+1} = \frac{a_n + \frac{4}{a_n}}{2}$$

converge? If yes, then what is the limit?

2.5.17. (5) Determine the limit of the following recursively defined sequence! $a_1 = 0,\ a_{n+1} = 1/(4 - a_n)\ (n = 1, 2, \ldots);$

2.5.18. (3) Let the sequence (a_n) be given by the following recursion: $a_1 = 0$, $a_{n+1} = \sqrt{a_n + 6}$. Prove that (a_n) is convergent and calculate its limit.

2.5.19. (4) Let $a_1 = 1$, $a_{n+1} = a_n + \dfrac{2}{a_n^2}$. Prove the existence of an $n \in \mathbb{N}$, for which $a_n \geq 10$.

Solution→

Related problem: 2.2.10

2.5.20. (2) Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 3}$. Prove that $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$.

2.5.21. (4) Let $a_1 = 1$,

$$a_{n+1} = a_n + \frac{1}{a_n^3}.$$

Is it true that $\exists n\ a_n > 10^{10}$?

2.6 The Number e

2.6.1. (3) Prove the following inequality:

$$\left(1 + \frac{1}{n}\right)^n \geq 2.$$

2.6.2. (5) Prove the following inequalities:

$$\left(\frac{n}{e}\right)^n < n! < e \cdot \left(\frac{n}{2}\right)^n.$$

2.6.3. (7) Prove the following inequalities.

$$0 < e - \left(1 + \frac{1}{n}\right)^n < \frac{3}{n}.$$

2.6.4. (5) Prove that

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2},$$

in other words the sequence $a_n = \left(1+\frac{1}{n}\right)^{n+1}$ is strictly monotone decreasing.

Solution→

2.6.5. (5) Prove that

$$n+1 < e^{1+\frac{1}{2}+\ldots+\frac{1}{n}} < 3n.$$

2.6.6. (9) Which one is greater? The number e or $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$?

2.6.7. (5) Prove that for all $n \in \mathbb{N}$ we have $n! > \left(\frac{n+1}{e}\right)^n$, and for $n \geq 7$ we have $n! < \frac{n^{n+1}}{e^n}$.

2.6.8. (6) Which one is the greater? $1000001^{1000000}$ or $1000000^{1000001}$.

1.2-3

2.6.9. (7) Find positive constants c_1, c_2 for which

$$c_1 \cdot \frac{n^{n+\frac{1}{2}}}{e^n} < n! < c_2 \cdot \frac{n^{n+\frac{1}{2}}}{e^n}$$

for all $n \in \mathbb{N}$.

2.6.10. (4) Calculate the limit of the sequence

$$a_n = \left(\frac{n+2}{n+1}\right)^n.$$

Solution→

2.6.11. (4) Calculate:

$$\lim \left(\frac{n+3}{n-1}\right)^{3n+8} = ?$$

2.6.12. (7) Verify that if $n \cdot a_n \to a$ and $b_n/n \to b$, then $(1+a_n)^{b_n} \to e^{ab}$.

2.6.13. (7) Prove for every sequence (a_n):

$$\liminf \left(1+\frac{1}{n}\right)^{a_n} = e^{\liminf \frac{a_n}{n}}.$$

2.7 Bolzano–Weierstrass Theorem and Cauchy Criterion

2.7.1. (4) The sequence a_n is monotone and it has a convergent subsequence. Does it imply that a_n is convergent?

Solution→

2.7.2. (5) Prove that if $|a_{n+1}-a_n| \le 2^{-n}$ for all n, then (a_n) is convergent.

2.7.3. (8) Prove that if the Bolzano–Weierstrass theorem holds in an ordered field, then it is isomorphic to $\mathbb{R}$.

2.7.4. (8) Prove that if in an Archimedean ordered field every Cauchy sequence is convergent, then every bounded set has a least upper bound.

2.7.5. (8) Prove that every Cauchy sequence is convergent, using the one-dimensional Helly theorem.

2.8 Infinite Sums: Introduction

2.8.1. (4)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$$

2.8.2. (5)

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n + \frac{1}{2}} = ?$$

2.8.3. (3) Convergent or divergent?

$$\sum \frac{n^{100}}{1.001^n}$$

2.8.4. (3) Convergent or divergent?

$$\sum \frac{1}{\sqrt{(2i-1)(2i+1)}}$$

2.8.5. (2)

$$\sum_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{3^i} \right) = ?$$

2.8.6. (5) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

Solution→

2.8.7. (2) Suppose that $\sum a_n$ is convergent. Show that $\lim(a_{n+1} + a_{n+1} + \ldots + a_{n^2}) = 0$.

2.8.8. (4) Find a sequence a_n such that $\sum a_n$ is convergent, and a_{n+1}/a_n is not bounded.

Solution→

2.8.9. (6) Convergent or divergent?

$$\sum \frac{(2k)!}{4^k (k!)^2}$$

2.8.10. (6) Convergent or divergent?

$$\sum \frac{(2k)!}{4^k (k!)^2} \frac{1}{2k+1}$$

2.8.11. (7) For which $z \in \mathbb{C}$ is the following sum convergent?

$$\sum z^n \qquad \sum \frac{z^n}{n} \qquad \sum \frac{z^n}{n^2}$$

2.8.12. (4) Convergent or divergent?

$$\frac{1000}{1} + \frac{1000 \cdot 1001}{1 \cdot 3} + \frac{1000 \cdot 1001 \cdot 1002}{1 \cdot 3 \cdot 5} + \dots$$

2.8.13. (3) Convergent or divergent?

$$a) \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \qquad\qquad b) \sum_{n=1}^{\infty} \frac{n^2}{(2+\frac{1}{n})^n}$$

2.8.14. (5) Convergent or divergent?

$$\sum_{n=1}^{\infty} (\sqrt[n]{e} - 1)$$

2.8.15. (5) Show that if $|a_{n+1} - a_n| < \frac{1}{n^2}$, then (a_n) is convergent.

Hint→

2.8.16. (7) $h_n := \sum_{i=1}^{n} \frac{1}{i}$. Prove that

$$\frac{1}{h_1^2} + \frac{1}{2h_2^2} + \dots + \frac{1}{nh_n^2} < 2.$$

2.8.17. (5) For which x and p is the sum

$$\sum \frac{x^n}{n^p}$$

convergent?

2.8.18. (4) Convergent or divergent?

$$\sum \frac{7^n}{\sqrt{n!}}$$

2.8.19. (4) For which x is the sum

$$\sum \frac{x^n}{a^n + b^n}$$

convergent?

2.8.20. (5) (a) Prove that if $\liminf \frac{\log \frac{1}{a_k}}{\log k} > 1$, then $\sum a_k$ is convergent.

(b) Prove that if $\limsup \frac{\log \frac{1}{a_k}}{\log k} < 1$, then $\sum a_k$ is divergent.

(c) Construct a sequence a_n such that $\frac{\log \frac{1}{a_k}}{\log k} \to 1$, and $\sum a_k$ convergent.

(d) Construct a sequence a_n such that $\frac{\log \frac{1}{a_k}}{\log k} \to 1$, and $\sum a_k$ divergent.

2.8.21. (4) For which x the sum

$$\sum \log\left(\frac{k+1}{k}\right) x^k$$

is convergent?

Chapter 3

Limit and Continuity of Real Functions

3.1 Global Properties of Real Functions

3.1.1. (2) Show that the following functions are injective on the given set H, and calculate the inverse.

1. $f(x) = 3x - 7,\ H = \mathbb{R}$;   2. $f(x) = x^2 + 3x - 6,\ H = [-3/2, \infty)$.

3.1.2. (2) Show that the following functions are injective on the given set H, and calculate the inverse.

1. $f(x) = \dfrac{x}{x+1},\ H = [-1, 1]$;   2. $f(x) = \dfrac{x}{|x|+1},\ H = \mathbb{R}$.

Solution→

3.1.3. (7) Find a function $f : [-1, 1] \to [-1, 1]$ such that $f(f(x)) = -x\ \forall x \in [-1, 1]$.

3.1.4. (4) Construct a non-constant periodic function with arbitrarily small periods.

3.1.5. (1) Find the inverse of $f(x) = \dfrac{2x-3}{3x-2}$ on $\mathbb{R} \setminus \{\frac{2}{3}\}$.

3.1.6. (2) Are the following functions injective on $[-1, 1]$?

a) $f(x) = \dfrac{x}{x^2+1}$,   b) $g(x) = \dfrac{x^2}{x^2+1}$.

Solution→

3.1.7. (2) Prove that all function $f:\mathbb{R}\to\mathbb{R}$ can be obtained as the sum of an even and an odd function.

3.1.8. (2) Let

$$f(x)=\begin{cases} x^3 & \text{if } x \text{ rational} \\ -x^3 & \text{if } x \text{ irrational.}\end{cases}$$

Does $f(x)$ have a unique inverse on $(-\infty,+\infty)$?

3.1.9. (4) Let $f(x)=\max\{x,1-x,2x-3\}$. Is it monotone, or convex?

3.1.10. (2) Prove that if f is strictly convex on the interval I, then every line intersects the graph of f in at most 2 points.

3.1.11. (1) Does there exist a function $f:(0,1)\to\mathbb{R}$ which is bounded, but has no maximum?

3.1.12. (2) Does there exist a function $f:[0,1]\to\mathbb{R}$ which is bounded, but has no maximum?

3.1.13. (4) Does there exist a monotone function f such that

1. $D(f)=[0,1]$, $R(f)=(0,1)$;
2. $D(f)=[0,1]$, $R(f)=[0,1]\cup[2,3]$;
3. $D(f)=[0,1]$, $R(f)=[0,1)\cup[2,3]$;
4. $D(f)=[0,1]$, $R(f)=[0,1)\cup(2,3]$?

3.1.14. (8) Does there exist a function which attains every real values on any interval?

3.1.15. (5) Prove that x^k is strictly convex on $[0,\infty)$, for all $k>1$ integer.

3.1.16. (3) Prove that if $a_1,...,a_n\geq 0$ and $k>1$ is an integer, then $\dfrac{a_1+...+a_n}{n}\leq\sqrt[k]{\dfrac{a_1^k+...+a_n^k}{n}}$.

3.1.17. (4) Prove that if $g:A\to B$ and $f:B\to C$ are convex, and f is monotone increasing, then $f\circ g$ is convex.

3.1.18. (4) Prove that if f is convex, then it can be obtained as the sum of a monotone increasing and a monotone decreasing function.

3.1.19. (7) Can we obtain the function x^2 as a sum of two periodic functions?

3.1.20. (10) Can we obtain the function x^2 as a sum of three periodic functions?

3.2 Continuity and Limits of Functions

3.2.1. (2) Find a good δ or L for $\varepsilon > 0$ or for K for the following functions.

1. $\lim\limits_{x\to 1+} (x^2+1)/(x-1)$, 2. $\lim\limits_{x\to\infty} \frac{\sin(x)}{\sqrt{x}}$.

3.2.2. (2) Determine the points of discontinuity of the following functions. What type of discontinuities are these?

1. $f(x) = \dfrac{x^3-1}{x-1}$, 2. $g(x) = \dfrac{x^2-1}{|x-1|}$,

3. $h_1(x) = x[\dfrac{1}{x}]$, 4. $h_2(x) = x^2[\dfrac{1}{x}]$.

3.2.3. (3) Determine the points of discontinuity of the following functions. What type of discontinuities are these?

1. $\dfrac{x^3-1}{(x-1)(x-2)(x-3)}$, 2. $\dfrac{1}{\left[\frac{1}{x}\right]}$.

3.2.4. (2) Determine the points of discontinuity of the following functions. What type of discontinuities are these?

a) $f(x) = \dfrac{x-2}{x^2-x-2}$, b) $g(x) = \operatorname{sgn}\left(\left\{\dfrac{1}{x}\right\}\right)$.

3.2.5. (2) Prove that $\lim\limits_{x\to a} f(x) = b \iff \lim\limits_{x\to a-0} f(x) = \lim\limits_{x\to a+0} f(x) = b$.

3.2.6. (1) Define: $\lim_{x\to a-} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = b$ and $\lim_{x\to -\infty} f(x) = +\infty$.

3.2.7. (1) Formulate the negation of $\lim_{x\to a} f(x) = +\infty$!

3.2.8. (1) Prove that the function $[x]$ is continuous in a if a is not an integer, and left-continuous if a is an integer.

3.2.9. (2) In which points are the following functions continuous?

1. $f(x) = \begin{cases} x & \text{if } \frac{1}{x} \in \mathbb{N} \\ 0 & \text{if } \frac{1}{x} \notin \mathbb{N} \end{cases}$, 2. $f(x) = \begin{cases} 3x+7 & \text{if } x \in \mathbb{Q} \\ 4x & \text{if } x \notin \mathbb{Q} \end{cases}$,

3. $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ cx & \text{if } x < 0 \end{cases}$.

3.2.10. (2) Where are they continuous?

1. Riemann-function, 2. $\sin \dfrac{1}{x}$, 3. $x \sin \dfrac{1}{x}$.

3.2.11. (2) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and $f(a) < g(a)$, then a has a neighborhood, where $f(x) < g(x)$.

3.2.12. (2) Let f be convex in $(-\infty, \infty)$ and assume that $\lim\limits_{x\to-\infty} f(x) = \infty$. Is it possible that $\lim\limits_{x\to\infty} f(x) = -\infty$?

3.2.13. (2) Let f be convex in $(-\infty, \infty)$ and assume that $\lim\limits_{x\to-\infty} f(x) = 0$. Is it possible that $\lim\limits_{x\to\infty} f(x) = -\infty$?

3.2.14. (1) Find a monotone function $f : [0, 1] \to [0, 1]$ with infinitely many points of discontinuity.

3.2.15. (3) The continuity of the function $f : \mathbb{R} \to \mathbb{R}$ at the point a is defined by:

$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.
Consider the following variations of this formula.
$(\forall\varepsilon > 0)(\forall\delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\varepsilon > 0)(\forall\delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\varepsilon > 0)(\exists\delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\forall\delta > 0)(\exists\varepsilon > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\delta > 0)(\forall\varepsilon > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.
Which properties of f are described by these formulas?

3.2.16. (1) Formulate the definition using the letters $\varepsilon, \delta, P, Q$ etc.:

$$\lim_{-\infty} f = 1; \quad \lim_{t\to t_0+0} s(t) = 0; \quad \lim_{\zeta\to-0} g(\zeta) = -\infty$$

$$\overline{\lim_{\vartheta\to-1}}\, h(\vartheta) = \infty; \quad \underline{\lim_{\xi\to-\infty}}\, u(\xi) = 2.$$

3.2.17. (1) Formulate the definition using the letters $\varepsilon, \delta, K, L$ etc.

$$\lim_{1} f = \infty; \quad \lim_{\eta \to \eta_0 -} s(\eta) = 2; \quad \lim_{x \to \infty} g(x) = -\infty;$$

$$\lim_{\omega \to \omega_0 -} s(\omega) = 2; \quad \underline{\lim}_{0+} g = 1; \quad \underline{\lim}_{\infty} h = 1.$$

3.2.18. (2) Prove that if f and g are continuous in the point a, then $\max(f, g)$ and $\min(f, g)$ are also continuous in the point a.

3.2.19. (2) Does the continuity of $g(x) = f(x^2)$ imply the continuity of $f(x)$?

3.2.20. (7) Assume that $g(x) = \lim_{t \to x} f(t)$ exists in every point. Prove that $g(x)$ is continuous.

Hint→

3.2.21. (3) Find an f and g such that $\lim_{x \to \alpha} f(x) = \beta$, $\lim_{x \to \beta} g(x) = \gamma$, but $\lim_{x \to \alpha} g(f(x)) \neq \gamma$.

3.2.22. (2) Can we extend $(\sqrt{x} - 1)/(x - 1)$ to $x = 1$ continuously?

3.2.23. (3) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is periodic and $\lim_{x \to \infty} f(x) = 0$, then f is identically zero.

3.2.24. (2) Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the preimage of every open set is open.

3.2.25. (7) Prove that if a function $\mathbb{R} \to \mathbb{R}$ is continuous in every rational point, then there is an irrational point as well where it is continuous.

3.2.26. (8) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(n \cdot a) \to 0$ for all $a > 0$. Prove that $\lim_{x \to \infty} f = 0$.

3.2.27. (2) In which points is the following function continuous?

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3.2.28. (2) In which points is the following function continuous?

$$f(x)=\begin{cases} x\sin\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

3.2.29. (2) In which points is the following function continuous?

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

3.2.30. (3) Prove that if $f:[0,1]\to\mathbb{R}$ is continuous, then $g(x):=\min\{f(x),0\}$ is also continuous.

3.2.31. (8) What is the cardinality of the set of continuous $\mathbb{R}\to\mathbb{R}$ functions?

3.2.32. (7) Is there an $\mathbb{R}\to\mathbb{R}$ function for which the limit is ∞ at every point?

3.2.33. (2)

$$\varliminf_{x\to\infty}\left(\{2x\}^2-4\{x\}^2\right)=? \qquad \varlimsup_{x\to\infty}\left(\{2x\}^2-4\{x\}^2\right)=?$$

3.3 Calculating Limits of Functions

3.3.1. (5)

$$\lim_{x\to 0}\frac{\sin x}{x}=? \qquad \lim_{x\to 0}\frac{e^x-1}{x}=?$$

3.3.2. (5)

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=?$$

3.3.3. (4)

$$\lim_{x\to 1}\frac{x+x^2+\ldots+x^n-n}{x-1}=?$$

3.3.4. (4)

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = ?$$

3.3.5. (4)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = ?$$

3.3.6. (4)

$$\lim_{x\to 3} \frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9} = ?$$

3.3.7. (4)

$$\lim_{x\to -2} \frac{\sqrt[3]{x-6}+2}{x^3+8} = ?$$

3.3.8. (4)

$$\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right) = ?$$

3.3.9. (4)

$$\lim_{x\to 0} (\sin\sqrt{x+1}-\sin\sqrt{x}) = ?$$

3.3.10. (4)

$$\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x} = ?$$

3.3.11. (4)

$$\lim_{x\to a} \frac{\sin(a+2x)-2\sin(a+x)+\sin(a)}{x^2} = ?$$

3.3.12. (4)

$$\lim_{x\to \frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{1-2\cos x} = ?$$

3.3.13. (5)

$$\lim_{x\to\frac{\pi}{6}} \frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1} =?$$

3.3.14. (5) Let

$$f(x)=\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$$

$\lim_{x\to 0} f(x) =?, \lim_{x\to 1} f(x) =?, \lim_{x\to\infty} f(x) =?$

3.3.15. (4)

$$\lim_{x\to-\infty} \frac{\log(1+e^x)}{x} =?$$

3.3.16. (5)

$$\lim_{x\to\frac{\pi}{6}} \frac{x^2\sin x-\frac{\pi^2}{72}}{x-\frac{\pi}{6}} =?$$

3.3.17. (6)

$$\lim_{x\to a} \left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} =?$$

3.3.18. (6)

$$\lim_{x\to 1/2} \left(\frac{x+2}{2x-1}\right)^{4x^2-1} =?$$

3.3.19. (6)

$$\lim_{x\to\infty} \frac{1+\sqrt{x}+\sqrt[3]{x}}{1+\sqrt[3]{x}+\sqrt[4]{x}} =?$$

3.3.20. (3)

$$(a)\ \lim_{x\to\infty} \frac{\sin e^x}{x} =? \qquad (b)\ \lim_{x\to\infty} \frac{x+\sin x}{\sqrt{x^2+1}} =?$$

3.3.21. (3) Calculate the limit at the given α of the following functions.

1. $f(x) = [x],\ \alpha = 2+0$; 2. $f(x) = \{x\}, \alpha = 2+0$;
3. $f(x) = \dfrac{x}{2x-1},\ \alpha = \infty$; 4. $f(x) = \dfrac{x}{2x-1},\ \alpha = \dfrac{1}{2}+0$;
5. $f(x) = \dfrac{x}{x^2-1},\ \alpha = \infty$; 6. $f(x) = \dfrac{x}{x^2-1},\ \alpha = 1-0$.
7. $f(x) = \sqrt{x+1}-\sqrt{x},\ \alpha = \infty$; 8. $\dfrac{\sqrt{x}+\sqrt[3]{x}}{x-\sqrt{x}},\ \alpha = \infty$;
9. $\dfrac{x^2+5x+6}{x^2+6x+5},\ \alpha = \infty$; 10. $2^{-[1/x]},\ \alpha = \infty$;
11. $\sqrt[3]{x^3+1}-x,\ \alpha = \infty$, 12. $x\{\frac{1}{x}\},\ \alpha = 0$,
13. $x[\frac{1}{x}],\ \alpha = 0$,

3.3.22. (3)

$$\lim_{x\to 2} \frac{\sqrt{x+2}-2}{\sqrt[3]{x+25}-3} = ?$$

3.3.23. (3) Calculate the following limits:

1. $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$
2. $\displaystyle\lim_{x\to 1} \frac{\sqrt[359]{x}-1}{\sqrt[5]{x}-1}$
3. $\displaystyle\lim_{x\to\infty} x\cdot\left[\sqrt{x^2+2x}-2\sqrt{x^2+x}+x\right]$
4. $\displaystyle\lim_{x\to\infty} x^{3/2}\cdot\left[\sqrt{x+2}+\sqrt{x}-2\sqrt{x+1}\right]$
5. $\displaystyle\lim_{x\to 1} \frac{(1-x)(1-\sqrt{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^n}$
6. $\displaystyle\lim_{x\to\infty} x+\sin(x)$

3.3.24. (3) Prove that

$$\lim_{x\to -\frac{d}{c}+} \frac{ax+b}{cx+d} = \begin{cases} \infty & \text{if } bc-ad>0 \\ -\infty & \text{if } bc-ad<0, \end{cases},$$

$$\lim_{x\to -\frac{d}{c}-} \frac{ax+b}{cx+d} = \begin{cases} -\infty & \text{if } bc-ad>0 \\ \infty & \text{if } bc-ad<0, \end{cases}$$

and

$$\lim_{x\to\pm\infty} \frac{ax+b}{cx+d} = \frac{a}{c}\ , \qquad (c\neq 0).$$

3.3.25. (3)

$$\lim_{x\to 1} \frac{x^{\sqrt{2}}-1}{x^{\pi}-1} = ? \qquad \lim_{x\to 7} \frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2} = ?$$

3.3.26. (4) Let $a>1$ and $k>0$. Prove that $\lim\limits_{x\to\infty} \dfrac{a^{\sqrt{x}}}{x^k} = \infty$.

3.3.27. (4)

$$\lim_{x\to\infty} \frac{\sqrt{4^x+x^3}-2^x}{(3/5)^x} = ?$$

3.3.28. (5)

$$\lim_{x\to 1} \left(\frac{n}{x^n-1} - \frac{m}{x^m-1} \right) = ?$$

3.3.29. (5)

$$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = ?$$

3.4 Continuous Functions on a Closed Bounded Interval

3.4.1. (3) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic. Does it imply that $f(x)$ is bounded?

3.4.2. (3) (Brouwer fixed-point theorem; 1-dimensional case.) All $f:[a,b]\to[a,b]$ continuous functions have a fixed point, i.e., an x, for which $f(x)=x$.

Solution→

3.4.3. (3) Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be continuous and $f(0) \geq g(0)$, $f(1) \leq g(1)$. Prove that there exists an $x \in [0,1]$, such that $f(x) = g(x)$.

3.4.4. (4) Let $f : [0,2] \to \mathbb{R}$ be continuous, $f(0) = f(2)$. Prove that the graph of f has a chord of length 1.

3.4.5. (5) Prove that if I is an interval (closed or not, bounded or not, might be a point) and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is also an interval.

3.4.6. (4) Prove that every polynomial of odd degree has a real root.

3.4.7. (4) Prove that the polynomial $x^3 - 3x^2 - x + 2$ has 3 real roots.

Solution→

3.4.8. (6) Prove that the continuous image of a compact set is compact.

3.4.9. (4) Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $x_1, x_2, \ldots, x_n \in [a,b]$, then there is a $c \in [a,b]$, for which $f(c) = \dfrac{f(x_1) + \cdots + f(x_n)}{n}$.

3.5 Uniformly Continuous Functions

3.5.1. (4) Are the following functions uniformly continuous?

a) x^2 on $(1,2)$,

b) $\sin x$ on $\mathbb{R}$,

c) $\sin \frac{1}{x}$ on$(0,\infty)$,

d) $1/x$ on $(0,2)$,

e) $\sqrt{x}$ on $(0,\infty)$.

3.5.2. (4) $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous. Does it imply that $f \cdot g$ is also uniformly continuous?

3.5.3. (4) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, then the function $f(x+5) - f(x)$ is bounded.

3.5.4. (5) Let $f : [0,1) \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous if and only if $\lim\limits_{1-0} f$ exists and is finite.

3.5.5. (8) Let $K \subset \mathbb{R}$. Prove that if all continuous $K \to \mathbb{R}$ functions are uniformly continuous, then K is compact.

3.6 Monotonity and Continuity

3.6.1. (2) Prove that if I is an interval and $f : I \to \mathbb{R}$ is continuous and injective, then it is strictly monotone.

3.6.2. (8) Is it true that if for the function $f : \mathbb{R} \to \mathbb{R}$ we have $\forall x \in \mathbb{R}$ $f(x-0) \le f(x) \le f(x+0)$, then f is monotone increasing?

3.7 Convexity and Continuity

3.7.1. (5) Prove that if $f : [a,b] \to \mathbb{R}$ is convex, then $\lim\limits_{a+0} f$ and $\lim\limits_{b-0} f$ exist and are finite, moreover $\lim\limits_{a+0} f \le f(a)$ and $\lim\limits_{b-0} f \le f(b)$.

3.7.2. (4) Is it true that if $f : \mathbb{R} \to \mathbb{R}$ is concave, then $\lim\limits_{-\infty} f < \infty$ or $\lim\limits_{\infty} f < \infty$?

3.7.3. (4) Is it true that if $f : \mathbb{R} \to \mathbb{R}$ is convex and $\lim\limits_{-\infty} f = -\infty$, then $\lim\limits_{\infty} f = \infty$?

3.7.4. (6) Prove that if f is weakly convex, then

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \le \frac{f(x_1) + \ldots + f(x_n)}{n}.$$

3.7.5. (4) Is it true that if $f : \mathbb{R} \to \mathbb{R}$ is concave and $\lim\limits_{-\infty} f$ is finite, then f is monotone decreasing?

3.7.6. (4) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is additive, then f^2 is weakly convex.

3.7.7. (4) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing and convex, then f^{-1} is concave on the interval $(\inf f, \sup f)$.

3.8 Exponential, Logarithm, and Power Functions

3.8.1. (7) Prove that if $f : \mathbb{R} \to (0, \infty)$ is continuous and for all $x, y \in \mathbb{R}$ the equality $f(x+y) = f(x) \cdot f(y)$ holds, then f is an exponential function.

3.8.2. (1) Which one is greater? $5^{\log_7 3}$ or $3^{\log_7 5}$?

3.8.3. (5) Suppose that $\varphi > 0$, and $\log \varphi$ is convex. Prove that φ is convex and show that the reverse implication does not hold.

3.8.4. (3) Prove that $\lim\limits_{x\to\infty} \dfrac{\log x}{x} = 0$ and $\lim\limits_{x\to+0} x \cdot \log x = 0$.

3.8.5. (4)

$$\lim_{x\to+0} x^x = ? \qquad \lim_{x\to+\infty} \sqrt[x]{x} = ?$$

3.8.6. (7) Prove that for the reals $0 < a < b$ the equality $a^b = b^a$ holds if and only if there is a positive number x for which $a = \left(1 + \frac{1}{x}\right)^x$ and $b = \left(1 + \frac{1}{x}\right)^{x+1}$.

3.8.7. (6)

$$\lim_{x\to+0} \left(1 + \frac{1}{x}\right)^x = ?$$

3.8.8. (6) Prove that if $0 < x$, $x \neq 1$, then $\log x < x - 1$.

3.8.9. (6) Prove that if $0 < x < 1$, then $\log(x) > 1 - \dfrac{1}{x}$.

3.8.10. (7) Find reals a, b such that for all $|x| < \frac{1}{2}$ we have $1 + x + ax^2 < e^x < 1 + x + bx^2$.

3.8.11. (7) Find reals a, b such that for all $|x| < \frac{1}{2}$ we have $x + ax^2 < \log(1 + x) < x + bx^2$.

3.8.12. (5)

$$\lim_{x\to-0} \left(1 + \frac{1}{x}\right)^x = ?$$

3.8.13. (4) Prove that if $x > 0$, $n \in \mathbb{N}$, then

$$e^x > 1 + \sum_{k=1}^{n} \frac{x^k}{k!}.$$

3.8.14. (4)

$$\lim_{x\to\infty} \frac{x^2 - \sqrt{x^3+1}}{\sqrt[3]{x^6+2} - x} = ?$$

3.8.15. (4)

$$\lim_{x\to\infty} \frac{\sqrt{2^x + 3^x} + 4^x}{\left(1 + \dfrac{1}{x}\right)^{x^2}} = ?$$

3.8.16. (4)

$$\lim_{x\to+0} e^{\log x/(\log|\log x|)} = ?$$

3.8.17. (5) Prove that $\log(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \leq (\log n) + 1.$

3.9 Trigonometric Functions and their Inverses

3.9.1. (5) (a) Prove that for $x \neq k\pi$ we have

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}.$$

(b)

$$\sin x + \sin 2x + \sin 3x + \ldots + \sin nx = ?$$

3.9.2. (5) Prove that for all non-negative integer n there are polynomials $T_n(x)$ and $U_n(x)$ of degree n, such that

$$T_n(\cos t) = \cos nt, \quad \text{and} \quad U_n(\cos t) = \frac{\sin(n+1)t}{\sin t},$$

and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{and} \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

(the so-called Chebishev polynomials.)

3.9.3. (6) (a) Express $\sin x$ and $\cos x$ using only $\tan x$.

(b) Express $\sin x$ and $\cos x$ using only $\tan \frac{x}{2}$.

(c) Express $\sin x$ and $\cos x$ using only $\cot \frac{x}{2}$.

Chapter 4

Differential Calculus and its Applications

4.1 The Notion of Differentiation

4.1.1. (2) Assume that $f : (a, b) \to \mathbb{R}$ is differentiable, $\lim_{x\to b} f(x) = \infty$. Does it imply that $\lim_{x\to b} f'(x) = \infty$?

4.1.2. (2)

$$\left(\sin\left(\frac{\sin x}{\sqrt{x}}\right)\right)' = ?$$

4.1.3. (3)

$$a)\ \ (x^x)' = ? \qquad\qquad b)\ \ \left((\sin x)^{\cos x}\right)' = ?$$

4.1.4. (3) Where is the function

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ -x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

differentiable?

4.1.5. (2) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, $\lim_{x\to\infty} f = 1$. Does it imply that $\lim_{x\to\infty} f' = 0$? And if we also know that $\lim_{x\to\infty} f'$ exists?

4.1.6. (3) Where is the function $\left(\{x\} - \frac{1}{2}\right)^2$ differentiable?

4.1.7. (3) Where is the function $f(x) = \dfrac{x}{|x|+1}$ differentiable? What is the derivative?

4.1.8. (3) Let $f(x) = x^2$ if $x \le 1$ and $f(x) = ax + b$ if $x > 1$. For which values of a and b will f be differentiable?

4.1.9. (4) Let $f(x) = x \cdot (x+1) \cdots (x+100)$, and let $g = f \circ f \circ f$. Calculate $g'(0)$.

4.1.10. (3) Prove that the function $f(x) = \sqrt{x}$ is differentiable for all $a > 0$ and $f'(a) = 1/(2\sqrt{a})$.

4.1.11. (3) Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Prove that if f is even, then f' is odd and vice versa.

4.1.12. (7) Let $[a, a+\delta) \subset D(f)$. Put the following quantities in increasing order:

$$\overline{f'_+}(a) \qquad \underline{f'_+}(a) \qquad \overline{\lim_{a+0}}\,\overline{f'} \qquad \overline{\lim_{a+0}}\,\underline{f'} \qquad \underline{\lim_{a+0}}\,\overline{f'} \qquad \underline{\lim_{a+0}}\,\underline{f'}$$

4.1.13. (2) Calculate the derivative:

$$-x; \qquad 3x^3 - 2x + 1; \qquad \frac{x^2+1}{x^3+2}; \qquad (x^{10} + x^2 + 1)^{100};$$

$$\frac{(x^3+1)^n}{(2+x)\left(x^3 + \dfrac{2}{x^2}\right)}$$

4.1.14. (2)

Calculate the derivative:

$$\frac{(x^2+1)^4(2-x)^8}{x^3+2} \cdot \frac{1 + \dfrac{1}{1+x}}{2-x}$$

4.1.15. (3)

Calculate the derivative:

$$\sin x^2 \qquad e^{\tan x} \qquad \log_3(\cot^2 x) \qquad \text{arc}\,\tan(x^2+1)$$

$$\sin\Big(\text{ar}\cosh\big(\text{arc}\cos(\log_5 x)\big)\Big)$$

4.1.16. (2) The following functions are derivatives. For which functions?

$$1+x+x^2; \qquad x+\frac{1}{x}; \qquad \frac{x^2}{(x^3+1)^2}$$

4.1.17. (3) $8x+\cos x$ is strictly monotone increasing. What is the derivative of its inverse in 1?

4.1.18. (10) Does there exists a monotone $\mathbb{R} \to \mathbb{R}$ function which is not differentiable at any point?

4.1.19. (4) Let $f(x) = x^2 \cdot \sin(1/x)$, $f(0) = 0$. Prove that f is differentiable everywhere.

4.1.20. (4) Prove that x^x is differentiable for all $x > 0$. Calculate the derivative!

4.1.21. (3) x^x is strictly monotone increasing in $[1, \infty)$. What is the derivative of its inverse in 27?

4.1.22. (3) $x^5 + x^2$ is strictly monotone increasing in $[1, \infty)$. What is the derivative of its inverse in 2?

4.1.23. (3) Prove that $x + \sin x$ is strictly monotone increasing in $[1, \infty)$. What is the derivative of its inverse in $1 + (\pi/2)$?

4.1.24. (4) Find a function $f(x)$ for which $f'(0) = 0$, and not differentiable at any other points.

4.1.25. (6) Prove that if $f'(x) \geq \frac{1}{100}$, then $\lim\limits_{x\to\infty} f(x) = \infty$.

4.1.26. (4) Prove that if $f'(x) = x^2$ for all x, then there is a constant c such that $f(x) = (x^3/3) + c$.

4.1.27. (5) Prove that if $f'(x) = f(x)$ for all x, then there is a constant c, such that $f(x) = c \cdot e^x$.

4.1.28. (4) Prove that if $f(a) = g(a)$ and for $x > a$ we have $f'(x) \geq g'(x)$, then $f(x) \geq g(x)$ for all $x > a$.

4.1.29. (3) Calculate the derivative of the following functions.

$$x^3; \qquad 2^x; \qquad \log_{1/2} x; \qquad \frac{1}{\sqrt{x}}; \qquad e^x + 3\log x \qquad x^2 3^x$$

$$\frac{\sin x}{x} \qquad x^3 e^x \cos x; \qquad x^3 \cdot \left(\frac{1}{2}\right)^x; \qquad \frac{x^2 \cdot \log x \cdot 3^x \cdot \cos x}{\sqrt{x} - \frac{3\sin x}{x^3}}.$$

4.1.30. (3) What is the derivative of the inverse function of $x^5 + x^3$ at the point -2?

4.1.31. (4) Find a function f such that $\lim_{x\to\infty} f'(x) = 0$, but $\lim_{x\to\infty} f(x) \neq 0$.

4.1.32. (4) Assume that 1. $x \cdot f(x)$, 2. $f(x^3)$, 3. $f^3(x)$
is differentiable at 0. Does it imply that $f(x)$ is differentiable at 0?

4.1.33. (3) Prove that if $f(a) = g(a)$ and $f(x) \leq g(x)$ in a neighborhood of a, then $f'(a) = g'(a)$.

4.1.34. (5) Calculate the derivative of the Chebishev polynomials at 1: $T_n'(1) =?$ $U_n'(1) =?$

4.1.35. (3) Calculate the derivative of the following functions.

$$x^2 e^{x^2 + \cos x^2} \qquad \log_{\coth^2 x + 1} \cot \frac{5^{\tan x}}{\cosh x} \qquad \frac{\frac{2^{\log x/2}}{x} + \operatorname{ar}\coth x}{\sqrt[3]{x} + \sqrt[5]{x}}$$

$$\frac{\frac{\tan x}{x^2+1} \cdot \frac{\sqrt{x} \cdot 10^x}{\log_3 x + x\cot x}}{(x+1)(x^2 + x^e)\cos x}$$

4.1.36. (4) Let

$$f(x) = \begin{cases} x + 2x^2 \cdot \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(0) > 1$, but f is not monotone increasing in any neighborhood of 0.

4.1.37. (3)

$$\left(f(x)^{g(x)}\right)' = ? \qquad \left(\log_{f(x)} g(x)\right)' = ?$$

4.1.38. (2) Calculate the derivative of both sides of the identity

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \qquad (x \neq 1).$$

4.1.39. (5) Is there a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = \infty$ for all x?

4.1.40. (6) Find an everywhere differentiable function with a non-continuous derivative!

Check the Darboux theorem for the derivative!

4.1.41. (5)

Is it true that if f is continuous in a and $\lim_{x \to a} f'(x) = \infty$, then $f'(a) = \infty$?

4.1.42. () Assume that $f : (a, b) \to \mathbb{R}$ is differentiable and $\lim_b f(x) = \infty$. Does it imply that $\lim_b f'(x) = \infty$?

4.1.43. (3)

Calculate the derivative!

1. $\sin\left(\frac{\sin x}{\sqrt{x}}\right)$, 2. x^x, 3. $(\sin x)^{\cos x}$.

4.1.44. (4) Suppose that f is differentiable and $|f'| < K$. Then f is uniformly continuous.

4.1.45. (4)

Prove that the graph of the function

$$f(x) = \begin{cases} x^x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is tangent to the y-axis.

4.1.46. (5)

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} = ?$$

4.1.47. (5) Prove that if f is differentiable at a, then

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

Show that the statement cannot be reversed.

4.1.1 Tangency

4.1.48. (3) In what angle do the graphs of the functions sin and cos intersect?

4.1.49. (4) Does the function $\sqrt[3]{\sin x}$ have a vertical tangent line?

4.1.50. (4) Prove that the line $y = mx + b$ is tangent to the graph of x^2 if and only if they intersect in one point.

4.1.51. (3) Which horizontal line is tangent to the graph of $2x^3 - 3x^2 + 8$?

4.1.52. (4) At which point is the x-axis tangent to the graph of $x^3 + px + q$?

4.1.53. (3) At what angle does the line $y = 2x$ intersect the graph of x^2?

4.1.54. (5) Prove that the graphs of $\sqrt{4a(a-x)}$ and $\sqrt{4b(b+x)}$ intersect each other perpendicularly.

4.1.55. (6) Prove that the graphs of $x^2 - y^2 = a$ and $xy = b$ intersect each other perpendicularly.

4.1.56. (6) Prove that the graphs of $ax = x^2 + y^2$ and $by = x^2 + y^2$ intersect each other perpendicularly.

4.1.57. (6) Prove that the graphs of $x^3 - 3xy^2 = a$ and $y^3 - 3x^2y = b$ intersect each other perpendicularly.

4.1.58. (4) At what angle do the graphs of 2^x and $(\pi - e)^x$ intersect?

4.2 Higher Order Derivatives

4.2.1. (5) Is it true that if $f'''(x) = f(x)$ for all $x \in \mathbb{R}$, then $f(x) = c \cdot e^x$ for some $c \in \mathbb{R}$?

4.2.2. (4) Is it true that if f is 7 times differentiable on $\mathbb{R}$, $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to\infty} f(x) = 3$, then f has an inflection point?

4.2.3. (6) Is it true that if f is 2 times differentiable at a, then

$$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a) \ ?$$

4.2.4. (6) Find a differentiable function f which is equal to $2x$ for $x \le 0$, and equal to $3x$ for $x \ge 1$. Is there a 2 times differentiable function? And a 271 times differentiable function?

4.2.5. (5) Calculate all derivatives of

$$f(x) = \frac{ax+b}{cx+d}.$$

4.2.6. (2) Let $f(x) = C_1 \cos x + C_2 \sin x$. $f''(x) + f(x) =?$

4.2.7. (2) Calculate the following derivatives:

1. $\left(e^{(x^3)}\right)^{(60)}(0)$, 2. $\left(e^{x^4}\right)^{(102)}(0)$, 3. $\left(e^{x^4}\right)^{(100)}(0)$.

4.2.8. (5) Assume that $f \in C^\infty(0,\infty)$, $\lim_{0+0} f = \lim_\infty f = 0$. Prove that $\exists \xi > 0$: $f''(\xi) = 0$.

4.2.9. (3) How many times is the function $|x|^3$ differentiable at 0?

4.2.10. (4) Find a function which is k times differentiable at 0 but not $k+1$ times.

4.2.11. (4) How many times is the function $|x|^\alpha$ differentiable at 0 if $\alpha > 0$?

4.2.12. (5) Assume that f and g are n times differentiable at the point a.

(a) Prove that fg is also n times differentiable at the point a.

(b) $(fg)^{(n)}(a) =?$

4.2.13. (5) Prove that

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

4.3 Local Properties and the Derivative

4.3.1. (5) (a) Prove that if f is convex, then the left and right derivatives exist at every point.

(b) Prove that if f is convex, then f'_+ is monotone increasing.

4.3.2. (2) Let $D(f) = [0,1]$, $f(x) = x^7(1-x)^9$. What are the zeroes of f'? What is the minimum and maximum of f?

4.3.3. (6) Prove that if $a \in (-1,1)$ is a local extremum of the Chebishev polynomial of second type U_n ($U_n(\cos t) = \dfrac{\sin(n+1)t}{\sin t}$), then

$$|U_n(a)| = \frac{n+1}{\sqrt{(n+1)^2(1-a^2)+a^2}}.$$

4.3.4. (4) Let

$$f(x) = \begin{cases} x^4 \cdot \left(2 + \sin\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f has a strict local maximum at 0, but f' does not change its sign at 0.

4.4 Mean Value Theorems

4.4.1. (4) Using the Lagrange mean value theorem prove that if f is differentiable on $\mathbb{R}$ and f' is bounded, then f is Lipschitz.

4.4.2. (5) Using the Lagrange mean value theorem prove that if $f'(a+0)$ exists, then $f'_+(a)$ also exists and they are equal.

4.4.3. (9) Let $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$ be real numbers. Show that

$$\det \begin{pmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_2 b_1} & e^{a_2 b_2} & \ldots & e^{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_n b_1} & e^{a_n b_2} & \ldots & e^{a_n b_n} \end{pmatrix} > 0.$$

(KöMaL A. 463., October 2008)

Solution→

4.4.1 Number of Roots

4.4.4. (3) Prove that the function $x^5 - 5x + 2$ has 3 real roots.

4.4.5. (3) Prove that the function $x^7 + 8x^2 + 5x - 23$ has at most 3 real roots.

4.4.6. (5) At most how many real roots does the function $x^{16} + ax + b$ have?

4.4.7. (4) For which values of k does the function $x^3 - 6x^2 + 9x + k$ have exactly one real root?

4.4.8. (8) At most how many real roots does the function $e^x + p(x)$ have if p is a polynomial of degree n?

4.5 Exercises for Extremal Values

4.5.1. (2) Which of the right circular cones inscribed into the unit sphere has the greatest volume?

4.5.2. (2) Calculate the extremal values of the following functions on the given interval!

1. $x^2 - x^4$; $[-2, 2]$; 2. $x - \arctan x$; $[-1, 1]$; 3. $x + e^{-x}$; $[-1, 1]$;
4. $x + x^{-2}$; $[1/10, 10]$; 5. $\arctan(1/x)$; $[1/10, 10]$; 6. $\cos x^2$; $[0, \pi]$;
7. $\sin(\sin x)$; $[-\pi/2, \pi/2]$; 8. $x \cdot e^{-x}$; $[-2, 2]$; 9. $x^n \cdot e^{-x}$; $[-2n, 2n]$;
10. $x - \log x$; $[1/2, 2]$; 11. $1/(1+\sin^2 x)$, $(0, \pi)$; 12. $\sqrt{1 - e^{-x^2}}$; $[-2, 2]$;
13. $x \cdot \sin(\log x)$; $[1, 100]$; 14. x^x; $(0, \infty)$; 15. $\sqrt[x]{x}$; $(0, \infty)$;
16. $(\log x)/x$; $(0, \infty)$; 17. $x \cdot \log x$; $(0, \infty)$; 18. $x^x \cdot (1-x)^{1-x}$; $(0, 1)$.

4.5.1 Inequalities, Estimates

4.5.3. (4) Prove that

$$\frac{\sin x + \sin y}{2} \le \sin\frac{x+y}{2} \qquad (x, y \in [0,\pi])\ !$$

4.5.4. (4) Prove that on the interval $(0, \pi/2)$ we have $\tan x > x + \frac{x^3}{3}$.

4.5.5. (6) Prove that for all $x > 0$ we have

$$\frac{x}{1+x} < \log(1+x) < x.$$

4.5.6. (4) Prove that for all $x \in [0,1]$ we have
1. $2^x \le 1 + x \le e^x$, 2. $\frac{2}{\pi}x \le \sin x \le x$.

4.5.7. (4) Prove that $|\arctan x - \arctan y| \le |x - y|$ for all x, y.

4.5.8. (5) Let $x < 0$ and n positive integer. Which one is the greater? e^x or $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!}$?

4.5.9. (9) Prove that if $a > 1$ and $0 < x < \frac{\pi}{a}$, then $\dfrac{\sin ax}{\sin x} < ae^{-\frac{a^2-1}{6}x^2}$.

4.5.10. (9) Prove that for all positive integer n and $x > 0$ we have

$$\frac{\binom{n}{0}}{\sqrt{x}} - \frac{\binom{n}{1}}{\sqrt{x+1}} + \frac{\binom{n}{2}}{\sqrt{x+2}} - \frac{\binom{n}{3}}{\sqrt{x+3}} + - \ldots + (-1)^n\frac{\binom{n}{n}}{\sqrt{x+n}} > 0.$$

4.5.11. (4) Prove that $\cos x \ge 1 - \dfrac{x^2}{2}$.

4.5.12. (5) Prove that

$$\cos x < e^{-x^2/2},$$

if $0 < x < \frac{\pi}{2}$.

4.5.13. (7) Let $|x| < \dfrac{\pi}{2}$. Which one is greater, $\dfrac{\sin x}{x}$ or $e^{-x^2/2}$?

4.5.14. (4) What is the range of the function $x \mapsto \dfrac{e^x}{x}$ $(x \in \mathbb{R} \setminus \{0\}$?

4.5.15. (10) Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients and $n \geq 2$, and suppose that the polynomial $(x-1)^{k+1}$ divides $p(x)$ with some positive integer k. Prove that

$$\sum_{\ell=0}^{n-1} |a_\ell| > 1 + \frac{2k^2}{n}.$$

CIIM 4, Guanajuato, Mexico, 2012

Solution→

4.5.16. (5) Let $0 < x, y < \pi$. Which one is greater: $\sin\sqrt{xy}$, or $\sqrt{\sin x \cdot \sin y}$?

4.6 Analysis of Differentiable Functions

4.6.1. (4) Analyze the following functions!
1. e^{-1/x^2}, 2. x^x (without convexity), 3. $x + e^{-x}$, 4. $\sin(\sin x)$, 5. $3x - x^3$, 6. $\frac{2-x^2}{1+x^4}$, 7. $\log(1+x^2)$, 8. $x^3 - 3x$, 9. $x^2 - x^4$, 10. $x - \arctan x$, 11. $x + e^{-x}$, 12. $x + x^{-2}$, 13. $\arctan(1/x)$, 14. $\cos x^2$, 15. $\sin(\sin x)$, 16. $\sin(1/x)$, 17. $x \cdot e^{-x}$, 18. $x - \log x$.

4.6.2. (4) Analyze the following functions!
1. $1/(1+\sin^2 x)$, 2. $\left(1+\frac{1}{x}\right)^x$, 3. $\left(1+\frac{1}{x}\right)^{x+1}$, 4. $\sqrt{1-e^{-x^2}}$, 5. x^x, 6. $\sqrt[x]{x}$, 7. $(\log x)/x$, 8. $x \cdot \log x$, 9. $x^x \cdot (1-x)^{1-x}$, 10. $\arctan x - \frac{1}{2}\log(1+x^2)$, 11. $\arctan x - \frac{x}{x+1}$, 12. $x^4/(1+x)^3$, 13. $e^x/(1+x)$, 14. $e^x/\sinh x$, 15. $e^{-x} \cdot \left[\dfrac{1-x^2}{2}\sin x - \dfrac{(1+x)^2}{2}\cos x\right]$.

4.6.3. (4) Analyze the following function:

$$a)\ \frac{2-x^2}{1+x^4} \qquad b)\ \log(1+x^2).$$

4.6.4. (4) Let $f(x) = x^n \cdot e^{-x}$. $f\big((0,\infty)\big) = ?$

4.6.5. (4) Analyze the following function: $\dfrac{e^x}{1-x^2}$.

4.6.6. (4) Analyze the following function: $\frac{\pi}{4}x - \operatorname{arc}\tan x$.

4.6.1 Convexity

4.6.7. (3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex, $f(5) = 12$ and $\alpha = \lim_{x\to\infty} f(x)$. What are the possible values of α?

4.6.8. (6) In how many points can the graphs of two convex functions intersect? And a convex and a concave?

4.6.9. (4) Find the maximal intervals for which the following functions are convex or concave.
1. e^x, 2. $\log x$, 3. $|x|$, 4. x^a $(a \in \mathbb{R})$, 5. a^x $(a > 0)$
6. $\sin x$.

4.6.10. (5) $f : (a, b) \to \mathbb{R}$ is convex, $\psi : f(a, b) \to \mathbb{R}$ is convex and monotone increasing. Prove that in this case $\psi \circ f$ is also convex.

4.6.11. (4) Is it true that the inverse of a convex function is concave?

4.7 The L'Hospital Rule

4.7.1. (3)

$$\lim_{x\to 0} \frac{\cos(x^2) - 1}{x} = ?$$

4.7.2. (3)

$$\lim_{x\to 0} \frac{\cos(xe^x) - \cos(xe^{-x})}{x^3} = ?$$

4.7.3. (3) Calculate the following limits using L'Hospital's rule!
1. $\lim_{x\to\pi/2} \frac{\cos x}{\frac{\pi}{2} - x}$, 2. $\lim_{x\to 0+} x^{\sqrt{x}}$.

4.7.4. (3) Calculate the following limits using L'Hospital's rule and also using the Taylor polynomial!

1. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$, 2. $\lim\limits_{x\to 0} \dfrac{\cos(x^2)-1}{x}$, 3. $\lim\limits_{x\to 0} \dfrac{\cos(xe^x)-\cos(xe^{-x})}{x^3}$,

4. $\lim\limits_{x=\infty} \dfrac{1+\sqrt{x}+\sqrt[3]{x}}{1+\sqrt[3]{x}+\sqrt[4]{x}}$, 5. $\lim\limits_{x\to 0} \dfrac{(1+x)^5-(1+5x)}{x^2+x^5}$, 6. $\lim\limits_{x\to 0} \dfrac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$,

7. $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x(1+x)}{x^3}$.

4.7.5. (2) Calculate the following limits using some known derivatives.

$$\lim_{x\to 0} \frac{\cos^3 x + e^x - 2}{x} \qquad \lim_{x\to 0} \frac{\sinh x}{\log_2(1+x)}$$

4.7.6. (3)

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = ? \quad \lim_{x\to 0} \frac{\log\cos ax}{\log\cosh bx} = ? \quad \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{x^{-2}} = ?$$

$$\lim_{x\to 1} \left((x-1)\tan\frac{\pi x}{2}\right) = ? \quad \lim_{x\to\infty} \frac{\sin\log x}{x} = ?$$

Can we use the L'Hospital rule? Can we use the definition of the derivative at 0 (or 1)?

4.7.7. (3)

$$\lim_{x\to 0} \frac{2e^x + e^{-x} - 3}{\sin 2x + x^2 + \sinh x} = ? \qquad \lim_{x\to 1} x^{\frac{1}{1-x}} = ?$$

$$\lim_{x\to 1} (2-x)^{\tan\frac{\pi x}{2}} = ? \qquad \lim_{x\to\infty} \frac{2x+\sin x}{2x-\cos x} = ?$$

Can we use the L'Hospital rule? Can we use the definition of the derivative at 0 (or 1)?

4.7.8. (4) Can we use the L'Hospital rule for $\frac{0}{\text{anything}}$ type limits?

4.7.9. (4) Assume that f, g are k times differentiable, $\lim\limits_{\infty} |g| = \infty$, $g^{(k)} \neq 0$ and $\lim\limits_{\infty} \frac{f^{(k)}}{g^{(k)}} = \beta$. Does it imply that $\lim\limits_{\infty} \frac{f}{g} = \beta$?

4.7.10. (4)

$$\lim_{x\to 0} \log_{(1-x^2)}(\cos bx) = ? \qquad \lim_{x\to 0} \left(\frac{1+e^x}{1+\cos x}\right)^{\cot x} = ?$$

$$\lim_{x\to 0} \frac{2\coth(x^2) - \cot(1-\cos x)}{\log(1+x) - \sin x} = ?$$

4.7.11. (4)

$$\lim_{x\to 1} (x-1)^{\log_x 2} = ? \qquad \lim_{x\to 0} (\cosh x)^{\cot^2 x} = ?$$

4.7.12. (5)

$$\lim_{x\to 0} \frac{\cot x - \frac{1}{x}}{3^x - \cosh x} = ?$$

4.7.13. (5)

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{e^x - 1}\right) = ?$$

4.7.14. (5)

$$\lim_{x\to 0} \frac{\coth x - \cot x}{\log(1+x) - x} = ?$$

4.8 Polynomial Approximation, Taylor Polynomial

4.8.1. (4) Calculate the Taylor expansion of arc tan.

4.8.2. (3) Calculate the Taylor expansion of e^x and $e^{(x^2)}$.

4.8.3. (2) Write the polynomial $1 + 3x + 5x^2 - 2x^3$ as linear combination of powers of $x + 1$.

4.8.4. (4)

$$\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = ?$$

4.8.5. (4)

$$\lim_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3} =?$$

4.8.6. (2) Calculate the degree 5 Taylor polynomial of $\log(\cos x)$.

4.8.7. (5) $A =?$, $B =?$ if $\cot x = \frac{1+Ax^2}{x+Bx^3} + o(x^4)$.
$\cot x - 1/x = \frac{(A-B)x}{1+Bx^2} + o(?)$

4.8.8. (4) Calculate the Taylor expansion at 0.

$$a)\ \frac{1}{1-x} \qquad b)\ \frac{1}{1+x} \qquad c)\ \frac{1}{1+2x} \qquad d)\ \frac{1}{3+4x} \qquad e)\ \frac{1}{2+x^2}$$

$$f)\ \frac{1}{\sqrt{1+x}}$$

4.8.9. (3) Calculate the degree 3 Taylor polynomial at 0:

$$\frac{(1+x)^{100}}{(1-2x)^{40}(1+2x)^{60}}$$

4.8.10. (3) Calculate the degree 3 Taylor polynomial at 0 for $\sin(\sin x)$.

4.8.11. (3) What is the leading term of $(1+x)^x - 1$?

4.8.12. (6) Prove that $\lim n\left(e - \left(1+\frac{1}{n}\right)^n\right) = \frac{e}{2}$.

4.8.13. (3) Calculate the degree $0, 1, 2, 3, 4$ and 5 Taylor polynomial at 1 for x^3!

4.8.14. (6) Prove that e is irrational!

4.8.15. (5) Calculate the Taylor expansion (at 0 if not specified):
1. $\sin x$; 2. $\cos x$; 3. $\arctan x$; 4. $\arcsin x$; 5. $\frac{1}{1-x^2}$;
6. $\frac{1}{1+x^2}$; 7. e^x; 8. e^{x^2}; 9. $x^3 e^{-x^2}$; 10. $1/x,\ a=1$;
11. $\sin^2 x$; 12. $\operatorname{arc} \sin x$.

4.8.16. (4) For which values of $a, b \in \mathbb{R}$ does the following identity hold

$$\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i} \binom{b}{k-i} \ ?$$

4.8.17. (2) Prove the binomial theorem using the binomial expansion!

4.8.18. (1) Prove that

$$\binom{-1/2}{k} = \frac{(-1)^k}{4^k} \binom{2k}{k} .$$

4.8.19. (5) Prove that for $x > 0$

$$\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + - \ldots - \frac{x^{4n+3}}{(4n+3)!} < \sin x < \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + - \ldots - \frac{x^{4n+1}}{(4n+1)!}$$

and

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \ldots - \frac{x^{4n+2}}{(4n+2)!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \ldots + \frac{x^{4n}}{(4n)!}.$$

4.8.20. (6) Prove that $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{x^k}{k!} = e^x$ for all $x \in \mathbb{R}$.

Chapter 5

The Riemann Integral and its Applications

5.0.1 The Indefinite Integral

5.0.1. (1)

$$\int \frac{\mathrm{d}x}{x+5} = ? \qquad \int \sqrt[3]{1-3x}\,\mathrm{d}x = ? \qquad \int (e^{-x} + e^{-2x+3})\,\mathrm{d}x = ?$$

5.0.2. (2)

$$\int \frac{\mathrm{d}x}{5+4x^2} = ? \qquad \int \left(\frac{1-x}{x}\right)^2 \mathrm{d}x = ? \qquad \int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,\mathrm{d}x = ?$$

5.0.3. (3)

$$\int xe^{-x}\,\mathrm{d}x = ? \qquad \int x^2 \log x\,\mathrm{d}x = ? \qquad \int \tanh^2 x\,\mathrm{d}x = ?$$

5.0.4. (4)

$$\int \sqrt{1-t^2}\,\mathrm{d}t = ? \qquad \int \sqrt{1+x^2}\,\mathrm{d}x = ? \qquad \int \frac{\mathrm{d}x}{\sin x} = ?$$

5.0.5. (5)

$$\int |x| \, \mathrm{d}x = ? \qquad \int |x^2 - 1| \, \mathrm{d}x = ? \qquad \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} \, \mathrm{d}x = ?$$

5.0.6. (4)

$$\int \frac{4x^5 - 5x^4 + 16x^3 - 19x^2 + 12x - 16}{(x-2)^2(x^4 + 4x^2 + 4)} \, \mathrm{d}x = ?$$

5.0.7. (4)

$$\int \frac{x^5 + 4x^4 + 12x^3 + 14x^2 + 15x + 12}{(x+2)(x^2+3)} \, \mathrm{d}x = ?$$

5.0.8. (4)

$$\int \frac{x^2}{\sqrt{1 + x + x^2}} \, \mathrm{d}x = ?$$

5.0.9. (5)

$$\int \sqrt{x^3 + x^4} \, \mathrm{d}x = ?$$

5.0.10. (5)

$$\int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} \, \mathrm{d}x = ?$$

5.0.11. (5)

$$\int \sin x \cdot \log(\tan x) \, \mathrm{d}x = ?$$

5.0.12. (4)

$$\int \frac{\mathrm{d}x}{1 + \sqrt{1 - 2x - x^2}} = ?$$

5.0.13. (4) $a, b \in \mathbb{R}$.

$$\int \frac{\mathrm{d}x}{a \sin x + b \cos x} = ?$$

5.0.2 Properties of the Derivative

5.0.14. (5) Find a non-continuous function with an antiderivative.

5.0.15. (4) Which of the following statements are true for any function $f : [a, b] \to \mathbb{R}$?

(a) If f is bounded, then it is Riemann-integrable.

(b) If f is bounded, then it has an antiderivative.

(c) If f has an antiderivative, then it is Riemann-integrable.

(d) If f has an antiderivative, then it is not Riemann-integrable.

(e) If f has an antiderivative, then it is bounded.

(f) f has an antiderivative if and only if its integral-function is an antiderivative.

(g) If f is integrable and its integral-function is differentiable, then the derivative of the integral-function coincides with f.

(h) If f is monotonically increasing, then its integral-function is convex.

(i) If the integral-function of f is convex, then f is monotonically increasing.

(j) If f satisfies the Intermediate Value Theorem, then it has an antiderivative.

5.1 The Definite Integral

5.1.1. (1) Use the definition of the Riemann integral to compute the integral over $[0, 1]$ of the function:

$a)$ x^2 $\qquad$ $b)$ $\begin{cases} 0 & x \leq 1/2 \\ 1 & x > 1/2 \end{cases}$ $\qquad$ $c)$ except finitely many points 0

5.1.2. (6) Let $0 < a < b$. Determine from the definition $\int_a^b x^m \, \mathrm{d}x$ by using an appropriate partition.

5.1.3. (3) State the necessary conditions and prove

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

5.1.4. (2) Is the following function Riemann-integrable on $[0,1]$?

$$f(x)=\begin{cases}1 & \text{if } x=\frac{1}{n},\ n=1,2,\dots\\ 0 & \text{otherwise}\end{cases}$$

5.1.5. (2) For a given ε find δ for which

$$\delta(F)<\delta \qquad \Rightarrow \qquad \left|\int_0^{10} e^x\,\mathrm{d}x\ -s_F(e^x)\right|<\varepsilon.$$

5.1.6. (3) Given ε find δ for which $|I-s_F|<\varepsilon$ if $\delta(F)<\delta$:

$a)$ $\sin x$ on $[0,2\pi]$; $\qquad$ $b)$ $f(x)=\begin{cases}0 & x=\frac{1}{n}, n=1,2,3,\dots\\ 1 & \text{otherwise on} \quad [0,1];\end{cases}$

$c)$ $\sin x\cup\{(0,0)\}$ on $[0,1]$.

5.1.7. (5) Is the Riemann function Riemann-integrable on $[0,1]$?

5.1.8. (5) Is the following function Riemann-integrable on $[0,1]$?

$$f(x):=\begin{cases}\frac{1}{\sqrt{q}} & x=\frac{p}{q},(p,q)=1,q>0\\ 0 & x \text{ irrational}\end{cases}$$

5.1.9. (5) Prove that if $\lim\limits_{\infty} f=A$, then $\lim\limits_{H\to\infty}\int_0^1 f(Hx)\,\mathrm{d}x=A$.

5.1.10. (1) Find the value of $\int_0^1 f$ if it exists,

$$f(x)=\begin{cases}1 & \text{if } x\in\left[\frac{1}{2^{2k+1}},\frac{1}{2^{2k}}\right],\ \ k=1,2,\dots\\ 0 & \text{otherwise.}\end{cases}$$

5.1.11. (4) If f is continuous and

$$\int_0^1 f(x)\,\mathrm{d}x=\int_0^1 xf(x)\,\mathrm{d}x=0,$$

then f has at least two different roots in $(0,1)$.

5.1.1 Inequalities for the Value of the Integral

5.1.12. (3) If f is bounded and concave down on $[a,b]$, then

$$(b-a)\frac{f(a)+f(b)}{2} \;\le\; \int_a^b f \;\le\; (b-a)f\Big(\frac{a+b}{2}\Big).$$

5.1.13. (5) Assume that $f:[0,\infty)\to\mathbb{R}$ is strictly increasing continuous and $f(0)=0$, $\lim_\infty f=\infty$. Let g be the inverse function f. Show that

$$xy \le \int_0^x f \;+\; \int_0^y g.$$

5.1.14. (3) Let $p,q>0$ and $1/p+1/q=1$. Show that for all $x,y\ge 0$

$$xy \le \frac{x^p}{p}+\frac{y^q}{q}.$$

5.1.15. (3) Prove the following:

(a) If $f,g:[a,b]\to\mathbb{R}$ are integrable, then $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$ (Schwarz inequality).

(b) If $f,g:[a,b]\to\mathbb{R}$ are integrable and $p,q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $\int_a^b fg \le \left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}$ (Hölder inequality).

5.1.16. (5) Prove that $xy \le (x+1)\log(x+1)-x+e^y-y-1$ holds for all pairs x,y of positive numbers.

5.2 Integral Calculus

5.2.1. (4)

$$a)\ \int_0^1 \frac{1}{\tan x+1}\,\mathrm{d}x = ? \qquad\qquad b)\ \int_0^1 x\ \arctan x\,\mathrm{d}x = ?$$

5.2.2. (4)

$$\int_0^{2\pi} \frac{1}{2+\cos x}\,\mathrm{d}x = ?$$

5.2.3. (3)

$$\int_0^3 x\cdot[x]\ \ \mathrm{d}x$$

5.2.4. (6)

$$\int_{0.1}^{0.2} \sqrt[\log\cosh\sin x]{1+\sinh^2\sin x}\ \ \mathrm{d}x = ?$$

5.2.5. (5)

$$\lim_{0+} \frac{\int_0^{\sin x}\sqrt{\tan t}\,\mathrm{d}t}{\int_0^{\tan x}\sqrt{\sin t}\ \ \mathrm{d}x} = ?$$

5.2.1 Connection between Integration and Differentiation

5.2.6. (4)

$$\left(\int_0^{x^4} e^{t^3}\sin t\ dt\right)' = ?$$

5.2.7. (5) Write down the second Taylor polynomial around 0 of the function

$$f(t) = \int_{t^2}^{-t^3-t} e^{x^2}\sin\sqrt{x}\ \ \mathrm{d}x.$$

5.3 Applications of the Integral Calculus

5.3.1. (4) Use Euler–Maclaurin summation to find

$$a)\ \sum_{k=1}^{n} k^5; \qquad b)\ \sum_{k=1}^{n} k^3(n-k)^3.$$

5.3.2. (4) How much work is required to elevate a mass from ground level to height h? To $h = \infty$?

5.3.3. (5) What curve is traced out by the centroids of the arc on the logarithmic spiral $r = a \cdot e^{m\varphi}$ $(r = a, \psi = 0) - P$ as P runs though all points on the spiral?

5.3.1 Calculating the Arclength

5.3.4. (4) Find the arclength of the arc on the parabola $y = x^2$ that lies above $[0, a]$.

5.3.5. (3) Find the arclength of the curve $r(\theta) = a + a\cos\theta$, $(\theta \in [\pi/4, \pi/4])$.

Hint→

5.3.6. (3) Prove that the logarithmic spiral $r = a \cdot e^{c\cdot\psi}$ $(\psi \in [0, \infty))$ has finite arclength.

5.4 Functions of Bounded Variation

5.4.1. (4) If $\gamma : [0, 1] \to \mathbb{R}^2$ is a continuous curve whose image contains $[0, 1] \times [0, 1]$, can γ be of bounded variation?

Hint→

5.4.2. (6) Prove that $f : [0, 1] \to \mathbb{R}$ is of bounded variation if and only if it is the sum of two monotonic functions.

Hint→

5.5 The Stieltjes integral

5.5.1. (2) Let f be continuous, $g(x) = \begin{cases} c & \text{if } x < \frac{a+b}{2} \\ d & \text{if } x > \frac{a+b}{2} \\ e & \text{if } x = \frac{a+b}{2} \end{cases}$.

$$\int_a^b f \, dg = ?$$

Hint→

5.5.2. (2) Let f be continuous.

$$\int_a^b f \; d[x] = ?$$

5.6 The Improper Integral

5.6.1. (6) Are the following improper integrals convergent? Absolute convergent?

$$a) \int_1^\infty \frac{\sin x}{x^2} \; \mathrm{d}x \qquad b) \int_1^\infty \frac{\sin x}{x} \; \mathrm{d}x \qquad c) \int_1^\infty \sin(x^2) \; \mathrm{d}x$$

5.6.2. (5) Prove that

$$\int_0^\infty x^n e^{-x} \; \mathrm{d}x = n!.$$

5.6.3. (2) Suppose that $\int_0^\infty |f|$ is convergent. Does it follow that $\lim_\infty f = 0$?

5.6.4. (5) Show that if f is uniformly continuous on $[2, \infty)$, then

$$\int_0^\infty \frac{f(x)}{x^2 \log^2 x} \; \mathrm{d}x$$

is convergent.

5.6.5. (3)

$$\lim_{0+0} x \cdot \int_x^1 \frac{\cos t}{t^2} \; \mathrm{d}t = ?$$

5.6.6. (2) Is the following integral convergent?

$$\int_0^3 \frac{\cos t}{t} \; \mathrm{d}t$$

Hint→

5.6.7. (5)

$$\int_0^{\pi/2} \log\cos x \ \ \mathrm{d}x = ?$$

5.6.8. (5) For what α is

$$\int_0^1 (x - \sin x)^\alpha \ \ \mathrm{d}x$$

convergent?

5.6.9. (7) Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ for which $\int_0^\infty f$ is convergent, but $\int_0^\infty f^2$ is divergent?

Chapter 6

Infinite Series

6.0.1. (1) Show that

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}.$$

6.0.2. (3) Prove

$$\frac{1}{n} \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n < 1.$$

6.0.3. (5) Prove that

$$a_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$$

is convergent.

6.0.4. (4)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = ?$$

6.0.5. (4)

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots = ?$$

6.0.6. (4)

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\ldots=?$$

6.0.7. (4)

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\ldots=?$$

6.0.8. (5) Let $u_n := \int_0^{1/n} \frac{\sqrt{x}}{1+x^2}\,\mathrm{d}x$. Is the series $\sum\limits_1^\infty u_n$ convergent?

6.0.9. (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots=?$$

6.0.10. (4)

$$\sum_{n=0}^{\infty}(n+1)q^n=?$$

6.0.11. (4) True or false?

(a) If $a_n \to 0$, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

(b) If $a_n \to 0$ and the partial sums $\sum\limits_{n=1}^{\infty} a_n$ are bounded, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

(c) If $\sum\limits_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

6.0.12. (4) Show that if $|a_n| < \frac{1}{n^2}$ for all positive integer n, then $\sum a_n$ satisfies the Cauchy criterion.

6.0.13. (8) Let $\sum\limits_{n=1}^{n} a_n$ be a divergent series with positive terms. Prove that there is a sequence c_n of positive numbers, such that $c_n \to 0$ as $n \to \infty$ and $\sum\limits_{n=1}^{n} (c_n \cdot a_n)$ still diverges.

6.0.14. (4)

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{4\cdot 5\cdot 6}+\ldots=?$$

6.0.15. (5)

$$\sum_{n=0}^{\infty} n^2 q^n = ?$$

6.0.16. (4) Assume that $a_n \le b_n \le c_n$ for all positive integer n. Show that if $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} c_n$ are convergent, then $\sum\limits_{n=1}^{\infty} b_n$ is also convergent.

6.0.17. (8) Let $\sum\limits_{n=1}^{n} a_n$ be a convergent series of positive terms. Prove that there is a sequence (c_n) such that $c_n \to \infty$ as $n \to \infty$ and for which $\sum\limits_{n=1}^{n} (c_n \cdot a_n)$ is still convergent.

6.0.18. (8) For $s > 1$ let $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$, $(p_1, p_2, p_3, \ldots) = (2, 3, 5, \ldots)$ be the sequence of primes in increasing order.

(a) Prove that $\lim\limits_{N\to\infty} \prod\limits_{n=1}^{N} \dfrac{1}{1 - \frac{1}{p_n^s}} = \zeta(s)$.

(b) Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{p_n} = \infty$.

(c) What is the order of magnitude of $\sum\limits_{n=1}^{\infty} \dfrac{1}{p_n^s}$ as $s \to 1 + 0$?

6.0.19. (9) For all $k \in \mathbb{N}$ let $\sum\limits_{n=1}^{\infty} a_n^{(k)}$ be a divergent series of positive terms. Prove that there is a sequence (c_n) of positive real numbers such that the series $\sum\limits_{n=1}^{\infty} (c_n \cdot a_n^{(k)})$ are all divergent.

6.0.20. (3) Determine whether the following series are convergent or divergent. In case of convergence determine whether convergence is absolute or conditional.

$$\sum_{n=1}^{\infty} \frac{1}{10n + \sqrt{n} + 1} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{[n/2]}}{\log(n+1)} \qquad \sum_{n=1}^{\infty} \frac{1}{n!}$$

6.0.21. (3) Determine whether the following series are convergent or divergent.

$$\sum e^{-n^2} \qquad \sum \frac{n^{10}}{3^n - 2^n} \qquad \sum \frac{1}{\sqrt{n(n+1)}} \qquad \sum n^2 e^{-\sqrt{n}}$$

$$\sum \left(n^{1/n^2} - 1\right) \qquad \sum \frac{\sqrt[n]{n} - 1}{\log^2 n}$$

6.0.22. (5) Assume that $a_n > 0$, $b_n > 0$ for all n and that $a_n/b_n \to 1$. Prove that $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent. Give an example when this fails if the assumption $a_n > 0, b_n > 0$ is removed.

6.0.23. (2) Prove that if $\sum a_n$ and $\sum b_n$ are absolutely convergent, then the following series are also absolutely convergent:

$$\sum (a_n + b_n) \qquad \sum \max(a_n, b_n) \qquad \sum \sqrt{a_n^2 + b_n^2}$$

6.0.24. (5) What are the root test, quotient test, Dirichlet-test, and Abel-test for improper integrals?

6.0.25. (3) Determine whether the following series are convergent or divergent. In case of convergence, determine whether the convergence is absolute or conditional.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \log(n+1)} \qquad \sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{2^{n^2}} \qquad \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}}$$

6.0.26. (4) Determine whether the following series are convergent or divergent.

$$\sum \left(1 - \frac{1}{n}\right)^n \qquad \sum \left(1 - \frac{1}{n}\right)^{n^2} \qquad \sum \left(\frac{n-1}{n+1}\right)^{\frac{n}{2}\log n + n \log\log n}$$

$$\sum \frac{n^{n+\frac{1}{n}}}{\left(n + \frac{1}{n}\right)^n}$$

6.0.27. (5) (a) Show that if $\overline{\lim}\left(|a_n|^{\frac{1}{\log n}}\right) < \dfrac{1}{e}$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) Show that if $a_n \geq 0$ and $\underline{\lim}\left(|a_n|^{\frac{1}{\log n}}\right) > \dfrac{1}{e}$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) Can any conclusions be made about the convergence of $\sum_{n=1}^{\infty} a_n$ if $a_n > 0$ and $\lim\left(|a_n|^{\frac{1}{\log n}}\right) = \dfrac{1}{e}$?

6.0.28. (6) Let $\sum a_{\varphi(n)}$ be a rearrangment of the conditionally convergent series $\sum a_n$. What can be the set of limit points of the set of the partial sums $\sum_{k=1}^{n} a_{\varphi(k)}$?

6.0.29. (7) Let $a_1, a_2, \ldots$ be a sequence of positive reals such that

$$\exists c > 0 \quad \forall x > 2 \quad \left|\{k : \; a_k < x\}\right| > c \frac{x}{\log x}.$$

(Primes for example satisfy this.) Show that $\sum \frac{1}{a_k} = \infty$.

6.0.30. (5) Prove the *Condensation lemma*: Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$. Then

$$\sum_{n=1}^{\infty} a_n \qquad \text{convergent} \iff \sum_{k=1}^{\infty} 2^k a_{2^k} \qquad \text{convergent.}$$

Solution→

6.0.31. (6) Convergent or divergent?

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Hint→

6.0.32. (6) Let $\varepsilon > 0$. Convergent or divergent?

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}}$$

Hint→

6.0.33. (4) For which $c \in \mathbb{R}$ is the series

$$\sum_{n=10}^{\infty} \frac{1}{n \cdot \log n \cdot (\log \log n)^c}$$

convergent?

6.0.34. (5) Using Dirichlet's criterion show that $\sum_{n=1}^{\infty} \frac{\sin(na)}{n}$ converges for all $a \in \mathbb{R}$.

6.0.35. (5) True or false?

(1) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (\sqrt[n]{2} \cdot a_n)$ is also convergent.

(2) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} (\sqrt[n]{2} \cdot a_n)$ is also divergent.

(3) If $\sum_{n=1}^{\infty} a_n$is convergent, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is also convergent.

(4) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is also divergent.

6.0.36. (5) Give examples of an absolutely convergent series $\sum_{n=0}^{\infty} a_n$ and conditionally convergent series $\sum_{n=0}^{\infty} b_n$ for which their Cauchy product is conditionally convergent.

6.0.37. (5) (Raabe criterion) Let $\sum_{n=1}^{\infty} a_n$ have positive terms.

(a) Prove that if $\liminf\ n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$, then the series is convergent.

(b) Prove that if $n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1$ for n large enough, then the series is divergent.

6.0.38. (10) For a sequence $A = (a_0, a_1, a_2, \ldots)$ of reals let

$$SA = (a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$$

be the sequence of its partial sums $a_0 + a_1 + a_2 + \ldots$. Can one find a non-zero sequence A for which the sequences A, SA, SSA, $SSSA$, $\ldots$ are all convergent?

Miklós Schweitzer memorial competition, 2007

Chapter 7

Sequences and Series of Functions

7.1 Convergence of Sequences of Functions

7.1.1. (3) For which values of x do the following sequences converge? On which intervals do they converge uniformly?

$$\sqrt[n]{|x|} \qquad \frac{x^n}{n!} \qquad x^n - x^{n+1} \qquad \left(1 + \frac{x}{n}\right)^n$$

7.1.2. (4) True or false?

(a) A pointwise limit of monotonic functions is monotonic.
(b) A pointwise limit of strictly monotonic functions is strictly monotonic.
(c) A pointwise limit of bounded functions is bounded.
(d) A pointwise limit of continuous functions is continuous.
(e) A pointwise limit of Lipschitz functions is Lipschitz.

7.1.3. (4) True or false?

(a) A uniform limit of monotonic functions is monotonic.
(b) A uniform limit of strictly monotonic functions is strictly monotonic.
(c) A uniform limit of bounded functions is bounded.
(d) A uniform limit of continuous functions is continuous.
(e) A uniform limit of Lipschitz functions is Lipschitz.

7.1.4. (3) A sequence of functions $f_1, f_2, \ldots : I \to \mathbb{R}$ is *uniformly bounded* if $\exists K \in \mathbb{R}\ \forall n \in \mathbb{N}\ \forall x \in I\ |f_n(x)| < K$.

Prove that the limit of a uniformly bounded sequence of functions is bounded.

7.1.5. (6) Prove that $\zeta(s)$ is infinitely differentiable on $(1, \infty)$.

7.1.6. (5) True or false? If a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ uniformly convergent on $[a, b] \cap \mathbb{Q}$, then it is uniformly convergent on $[a, b]$.

7.1.7. (9) True or false? From a sequence of uniformly bounded continuous functions $f_n : [a, b] \to \mathbb{R}$ one can select a uniformly convergent subsequence.

7.1.8. (3) For which values of x do the following sequences converge? On which intervals do they converge uniformly?

$$\frac{x^n}{1 + x^n} \qquad \sqrt[n]{1 + x^{2n}} \qquad \sqrt{x^2 + \frac{1}{n}}$$

7.1.9. (4) True or false?
(a) A pointwise limit of convex functions is convex.
(b) A pointwise limit of strictly convex functions is strictly convex.
(c) A pointwise limit of Riemann-integrable functions is Riemann-integrable.
(d) A pointwise limit of differentiable functions is differentiable.

7.1.10. (4) True or false?
(a) A uniform limit of convex functions is convex.
(b) A uniform limit of strictly convex functions is strictly convex.
(c) A uniform limit of Riemann-integrable functions is Riemann-integrable.
(d) A uniform limit of differentiable functions is differentiable.

7.1.11. (5) A sequence of functions $f_1, f_2, \ldots : I \to \mathbb{R}$ is *uniformly Lipschitz* if $\exists K \in \mathbb{R}\, \forall n \in \mathbb{N}\, \forall x, y \in I\; |f_n(x) - f_n(y)| \leq K|x - y|$. Prove that a pointwise limit of a sequence of uniformly Lipschitz functions is Lipschitz.

7.1.12. (7) Prove that a uniformly bounded and uniformly Lipschitz sequence of functions has a uniformly convergent subsequence.

7.1.13. (7) Prove that if $(f_n : H \to \mathbb{R})$ is uniformly convergent on all countable subsets of H, then it is uniformly convergent on H.

7.1.14. (5) True or false? If $f_1, f_2, \ldots$ is a sequence of continuous non-negative functions, then $F(x) = \inf\{f_1(x), f_2(x), \ldots\}$ is also continuous.

7.1.15. (9) True or false? If H is a non-empty bounded and closed subset of $C[a,b]$ and $f : H \to \mathbb{R}$ is a continuous map, then f has a maximum.

7.1.16. (9) Is the Baire theorem true for $C[a,b]$? That is, decide whether $C[a,b]$ can be presented as a union of countably many nowhere dense subsets.

7.2 Convergence of Series of Functions

7.2.1. (8) Show that if $\sum_{n=1}^{\infty} f_n$ converges uniformly on the set H after any rearrangment of the terms, then $\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent.

7.2.2. (4) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x}{x^2+1}\right)^n$ convergent? For which values is it absolutely convergent?

7.2.3. (4) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots (2n)} \left(\frac{2x}{x^2+1}\right)^n$ convergent? For which values is it absolutely convergent?

7.2.4. (4) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+3^{2n}}{2^n} x^n(1-x)^n$ convergent? For which values is it absolutely convergent?

7.2.5. (3) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$ convergent? For which values is it absolutely convergent?

7.2.6. (3) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1+x^{2n}}$ convergent? For which values is it absolutely convergent?

7.2.7. (4) For which values is the series $\displaystyle\sum_{n=1}^{\infty} ne^{-nx}$ convergent? For which values is it absolutely convergent?

7.2.8. (4) For which values is the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cos^n x}{n^2}$ convergent? For which values is it absolutely convergent?

7.2.9. (5) For which values is the series $\sum_{n=1}^{\infty}\left[\frac{x(x+n)}{n}\right]^n$ convergent? For which values is it absolutely convergent?

7.2.10. (7) Prove that if the Laurent series $\sum_{n=-\infty}^{\infty} a_n x^n$ converges at $x = r$ and $x = R$, $(0 < r < R)$ then it converges for all $x \in [r, R]$.

7.2.11. (5) For which x is

$$\sum_{n=-\infty}^{\infty} \frac{n}{a^{|n|}} x^n$$

convergent? Which is the value of the sum?

7.2.12. (6) Let $(x)_n = x(x-1)\dots(x-(n-1))$. At which points do the following Newton-type series converge and converge uniformly?

$$\sum_{n=1}^{\infty} \frac{(x)_n}{n!}; \qquad \sum_{n=1}^{\infty} \frac{1}{n^p}\frac{(x)_n}{n!}$$

where $p \in \mathbb{R}$.

7.2.13. (6) Assume that $f_n(x)$ are monotonic on $[a, b]$, and that

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely for $x = a$ and $x = b$. Show that the series converges absolutely and uniformly on $[a, b]$.

7.2.14. (7) Assume that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x - a_n}$$

converges on any closed interval that does not contain any of the $a_n (n = 1, 2, \dots)$. Is the convergence absolute? Is it uniform?

7.2.15. (7) Assume that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x}$$

converges for $x = x_0$. Prove that it converges for any $x > x_0$.

7.2.16. (7) Construct a series of functions that is both uniformly convergent and absolutely convergent but not uniformly absolute convergent.

7.2.17. (5) Give an example of non-negative uniformly convergent series, for which the Weierstrass criterion is not applicable.

7.3 Taylor and Power Series

7.3.1. (4) Determine the Taylor series of the function at the given point.

(a) $\dfrac{1}{1-x}$ at 0;
(b) $\dfrac{1}{x^2}$ at 3;
(c) $\log x$ at 5 körül;
(d) $\sin x$ at $\dfrac{\pi}{3}$;
(e) $\log(x^2-1)$ at 2;
(f) $\operatorname{ar\,sinh} x^2$ at 0;
(g) $\operatorname{ar\,coth} x$ at 2.
Give intervals where the Taylor series converges to the function.

7.3.2. (7) Construct an infinitely differentiable function f whose Taylor series around 0 converges everywhere but the limit equals $f(x)$ if and only if $x \in [-1,1]$.

7.3.3. (3) Determine the radius of convergence of the following series.

$$\sum n^{99}x^n \qquad \sum \left(1+\frac{1}{n}\right)^{n^2} x^n \qquad \sum n!x^{n^2}$$

7.3.4. (1) By the binomial theorem $(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ if $|x|<1$. Which identities result in the $\alpha=-1$ and $\alpha=-2$ cases?

7.3.5. (6)

$$\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+-\ldots=? \qquad \sum_{k=0}^{\infty}\frac{(-1)^k}{4k+1}=?$$

7.3.6. (6)

$$\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}-\frac{1}{3k+2}\right)=?$$

7.3.7. (6) Let $c_0 = 1$ and $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$. (Catalan numbers.) Define $G(x) = \sum_{n=0}^{\infty} c_n x^n$ the so-called generating function of the Catalan numbers.

(a) Prove that G converges in a neigborhood of 0.

(b) Prove that in the (non-empty) interior of the convergence interval $G(x) = xG^2(x) + 1$.

(c) Using b) determine G and c_n explicitely.

7.3.8. (8) Let p_n be the number of partitions of the number n into different parts. (For example $p_0 = 1$ and $p_6 = 4$, because $6 = 5+1 = 4+2 = 3+2+1$.) Using the generating series $P(x) = \sum_{n=0}^{\infty} p_n x^n$ find an upper bound for p_n.

7.3.9. (5) Determine the Taylor series of $\operatorname{ar}\tanh x$ around $a = 1/2$. For which x do the series equal the original function?

7.3.10. (6)

$$\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}+\frac{1}{3k+2}-\frac{2}{3k+3}\right)=?$$

7.3.11. (5) (a) For which real values of c will the series $\sum_{n=1}^{\infty}\Big(n^c \cdot \cos(nx)\Big)$ converge on $\mathbb{R}$?

(b) For which real values of c will the series $\sum_{n=1}^{\infty}\Big(n^c \cdot \sin(nx)\Big)$ converge uniformly on $\mathbb{R}$?

7.3.12. (2) For which $c \in \mathbb{R}$

$$\sum_{k=0}^{\infty}\binom{c}{k} = 2^c?$$

Chapter 8

Differentiability in Higher Dimensions

8.1 Real Valued Functions of Several Variables

8.1.1 Topology of the n-dimensional Space

8.1.1. (2) Find the interior, boundary and closure of the set

$$A = \left\{ \left(x, \sin \frac{1}{x} \right) \mid x > 0 \right\} \subset \mathbb{R}^2.$$

8.1.2. (5) True or false?

- a) $A \subset B \;\Rightarrow\; \operatorname{int} A \subset \operatorname{int} B$;
- b) $\operatorname{int}\operatorname{int} A = \operatorname{int} A$;
- c) $\partial \operatorname{int} A = \partial A$;
- d) $\overline{\operatorname{int} A} = \overline{A}$;
- e) $\overline{\overline{A}} = \overline{A}$;
- f) $\operatorname{int}(\overline{A}) = \operatorname{int} A$;
- g) $\partial \overline{A} = \partial A$

8.1.3. (5) Prove that $\overline{H}$ is the smallest closed set containing H.

8.1.4. (4) $\overline{H} = \{\ y \mid \exists x_n \in H$ sequence, for which $x_n \to y\ \}$.

8.1.5. (4) Show that
a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

8.1.6. (1) Prove that if p is a limit point of $E \subset \mathbb{R}^d$, then all neighborhoods of p contain infinitely many points of E.

8.1.7. (5) Show that for all $H \subset \mathbb{R}^d$ $\partial\partial H \subset \partial H$. Give an example when the inclusion is proper.

8.1.8. (6) Let $x \in \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be closed. Prove that there is $a \in A$ for which $|x - a| = d(x, A)$, where

$$d(x, A) := \inf\{|x - b| : b \in A\}$$

is the distance of x from A.

8.1.9. (6) Let $A \subset \mathbb{R}^d$ be closed such that its diameter

$$\operatorname{diam}(A) := \sup\{|x - y| : x, y \in A\}$$

is d. Prove that there are $a, b \in A$ whose distance is d.

8.1.10. (1) Determine the interior, exterior and boundary of the following sets. What is the boundary of the boundaries?

$$\big\{(x, y) \in \mathbb{R}^2 :\ x, y > 0,\ x+y < 1\big\}; \qquad \bigcup_{n=1}^{\infty} \left\{(x, y) \in \mathbb{R}^2 :\ x = 1/n,\ |y| < \frac{1}{n}\right\}$$

8.1.11. (5) For any subset A of a metric space show that

$$\operatorname{int}\operatorname{int} A = \operatorname{int} A; \quad \operatorname{int}\operatorname{ext} A = \operatorname{ext} A.$$

8.1.12. (6) Prove that if K is such a subset of a metric space that from all covers of K by open balls contain a finite subcover, then K is compact.

8.1.13. (8) Prove that if K is a compact subset of a metric space, then K is bounded and closed.

8.1.14. (5) Is there an $A \subset \mathbb{R}$ for which $\partial A, \partial\partial A, \partial\partial\partial A, \dots$ are all different?

8.1.15. (5) Prove that for any A, B subset of a metric space

$$\partial(A \cup B) \subset \partial A \cup \partial B;$$

$$\partial(A \cap B) \subset \partial A \cup \partial B.$$

Is it true that

$$\big(\partial(A \cup B)\big) \cup \big(\partial(A \cap B)\big) = \partial A \cup \partial B?$$

8.1.16. (6) (a) Prove that for any subset A of a metric space $\partial(\operatorname{int} A) \subset \partial A$ and $\partial(\operatorname{ext} A) \subset \partial A$.

(b) Is it true that $\partial(\operatorname{int} A) = \partial(\operatorname{ext} A)$?

8.1.17. (5) Prove that in a metric space the boundary of any set is closed.

8.1.18. (6) Prove that if K is a compact subset of a metric space, then all closed subsets of K are compact.

8.1.19. (1) Prove that in any metric space the cardinality of open and closed sets is the same.

8.1.20. (8) Prove that if in a metric space every bounded, closed set is compact, then the space is complete.

8.1.21. (9) (a) Prove that if the Bolzano–Weierstrass theorem is true in a metric space, then the space is complete.

(b) Give an example for a metric space that is complete but for which the Bolzano–Weierstrass theorem is not true.

8.1.22. (8) Prove that $\mathbb{R}^p$ has continuum many open (closed) subsets.

8.1.23. (9) A subset of $\mathbb{R}^p$ is "G_δ" if it is the intersection of countably many open sets. A chain H is a set of subsets of $\mathbb{R}^p$ such that from any two sets in H one is contained by the other. Prove that the intersection of any chain of open sets is G_δ.

8.1.24. (5) Collect as many descriptions of open and closed sets as you can.

8.1.25. (5) Prove that in $\mathbb{R}^p$ every closed interval $[a,b]$ is connected, that is, if $[a,b] \subset (A \cup B)$, then $[a,b] \subset A$ or $[a,b] \subset B$.

8.1.26. (6) Prove that $\mathbb{R}^p$ satisfies the Baire category theorem.

8.1.27. (9) Prove Helly's theorem:

(a) If $F_1, \ldots, F_n \subset \mathbb{R}^p$ are convex, and any $(p+1)$ among them have a common point, then the F_i-s have a common point.

(b) If $F_i \subset \mathbb{R}^p$ $(i \in I)$ are convex and compact and any $(p+1)$ among them have a common point, then the F_i-s have a common point.

8.1.28. (9) Show that the unit ball of $C[a,b]$ (with the maximum norm) is not compact.

8.1.29. (10) Is it true that the intersection of a chain of G_δ sets is G_δ?

8.1.30. (9)

8.1.2 Limits and Continuity in $\mathbb{R}^n$

8.1.31. (4) $\lim_{(0,0)}(x^2+y^2)^{x^2y^2} =?$

Answer→

8.1.32. (8) A norm on $\mathbb{R}^p$ is a function $||.|| : \mathbb{R}^p \to \mathbb{R}$ that satisfies

(a) $||x|| \geq 0$ and $||x|| = 0$ if and only $x = 0$;
(b) $||x+y|| \leq ||x|| + ||y||$;
(c) $||c \cdot x|| = |c| \cdot ||x||$ for all $c \in \mathbb{R}$, $x \in \mathbb{R}^n$.
Define the following norm on $\mathbb{R}^p$:

$$||x||_\alpha = \left(\sum_{i=1}^{p} |x_i|^\alpha\right)^{1/\alpha} \quad (1 \leq \alpha < \infty); \qquad ||x||_\infty = \max_{1\leq i\leq p} |x_i|.$$

(a) Prove that these are norms.
(b) Why do we need $1 \leq \alpha$?
(c) Prove that for all $x \in \mathbb{R}^p$

$$\lim_{\alpha\to\infty} ||x||_\alpha = ||x||_\infty.$$

(d) Show that

$$\forall \alpha, \beta \in [1,\infty) \cup \{\infty\}\ \exists c_1, c_2 > 0\ \forall x \in \mathbb{R}^p\ c_1||x||_\alpha < ||x||_\beta < c_2||x||_\alpha.$$

(e) Prove that any two norms are equivalent if $||.||$ and $||.||'$ are two norms, then there are $c_1, c_2 > 0$ such that $c_1||x|| \leq ||x||' \leq c_2||x||$.

8.1.33. (1) Prove that the map $(x,y) \mapsto x+y$ is continuous. Find δ for $\varepsilon = 10^{-3}$ at the point $(1,2)$.

8.1.34. (3) For what $\alpha \in \mathbb{R}$ is

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^\alpha} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$?

8.1.35. (5) Let $A \subset \mathbb{R}^p$ and $f : A \to \mathbb{R}$. Let $B \subset \mathbb{R}^p$ be the set of points where f has a limit at $b \in B$, and let $g(b) = \lim_{x \to b,\, x \in A} f(x)$. Prove that g is continuous on B.

8.1.36. (6) Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ and all sections $f_{x=a}$ are continuous and all sections $f_{y=b}$ are monotonic and continuous. Prove that f is continuous.

8.1.37. (7) Prove that if $K \subset \mathbb{R}^p$ and all continuous functions on K are bounded, then K is compact.

8.1.38. (1) Prove that $(x,y) \mapsto xy$ is continuous. Find δ for $\varepsilon = 10^{-3}$ at the point $(1,2)$.

8.1.39. (4) Find $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^p \to \mathbb{R}$ for which $\lim_{\mathbf{0}} g = 0$ and $\lim_{0} f = 0$ but $\lim_{\mathbf{0}} (f \circ g) \neq 0$.

8.1.40. (5)

$$\lim_{(x,y)\to(0,0)} \frac{\cos x + \cos y - 2}{x^2+y^2} = ?$$

For a given ε find δ.

8.1.41. (3) Prove that $f : \mathbb{R}^p \to \mathbb{R}$ is continuous if and only if the preimage of any open set is open.

8.1.42. (5) Does $\dfrac{\sin x - \sin y}{x-y}$ have a limit at the origin relative to the set $\{(x,y) : \ x \neq y\}$?

Can this function be extended continuously to the whole plane?

8.1.43. (1) $x_0 \in \mathbb{R}^p$. $f : \mathbb{R}^p \to \mathbb{R}$, $x \mapsto |x - x_0|$. Prove that f is continuous.

8.1.44. (4) For what $a > 0$ is $\dfrac{x^2 y}{(x^2+3y^2)^a}$ continuous at the origin?

8.1.45. (3) Let $A \subset \mathbb{R}^p$, $A \neq \emptyset$ and define $f : \mathbb{R}^p \to \mathbb{R}$,

$$f(x) := \inf\{\ |x-y| \mid y \in A\ \}.$$

Prove that f is continuous. Prove that

$$f(x) = 0 \qquad \Leftrightarrow \qquad x \in \overline{A}.$$

8.1.46. (8) Construct a Peano-curve, a continuous and surjective map from $[0,1]$ to $[0,1]^2$ and to $[0,1]^3$.

8.1.3 Differentiation in $\mathbb{R}^n$

8.1.47. (1) Is xy ($\mathbb{R}^2 \to \mathbb{R}$) differentiable? What is the derivative?

8.1.48. (2)

$$g(t) = \begin{cases} t^2 & \text{if } t \geq 0 \\ -t^2 & \text{if } t < 0 \end{cases}$$

At what points is $f(x,y) := g(x) + g(y)$ differentiable?

8.1.49. (2) Sketch the level curves of $f(x,y) = e^{\frac{2x}{x^2+y^2}}$. Given (x_0, y_0) in which direction does f grow fastest?

8.1.50. (3) At which points is $||\ .\ ||_1 := \sum |x_i|$ differentiable?

8.1.51. (3) Let $1 < p < \infty$. At which points is the $||\ .\ ||_p := (\sum |x_i|^p)^{1/p}$ function differentiable?

8.1.52. (7) Give a function $f : \mathbb{R}^2 \to \mathbb{R}$ for which all directional derivatives exist at $(0,0)$ but which is not differentiable at $(0,0)$.

8.1.53. (5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the distance of (x,y) from the interval $I := [0,1] \times \{0\}$. At which points is f differentiable? Twice differentiable?

8.1.54. (2) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable with derivative $(f(x,y), g(x,y))$. What is the derivative of $F(\sin t, \cos t)$?

8.1.55. (1) $f(x,y) = x^2 + y^3$, $g(x,y) = x^2 + y^4$. Calculate the first and second differentials at $(0,0)$.

8.1.56. (4)

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{otherwise} \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad \frac{\partial^2 f}{\partial y \partial x}(0,0) = ? \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = ?$$

8.1.57. (4) Is the function $(x,y) \mapsto \arcsin \frac{x}{y}$ uniformly continuous?

8.1.58. (2) Let $f(x,y) = \log\sqrt{(x-a)^2 + (y-b)^2}$. Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

8.1.59. (3)

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{otherwise} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is differentiable everywhere but not continuously.

8.1.60. (3) Let $g(t) = \operatorname{sgn}(t) \cdot t^2$. Show that $f(x,y) = g(x) + g(y)$ is everywhere differentiable but is not twice differentiable along the two axes.

8.1.61. (3) Show that $(x - y^2)(2x - y^2)$ has no local minimum at $(0,0)$ even though it has a local minimum along any lines through $(0,0)$.

8.1.62. (2) $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth. Give a normal vector of the graph of $z = f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

8.1.63. (3) Find the minimum and maximum of $x^3 + x^2 - xy$ on $[0,1] \times [0,1]$.

8.1.64. (3) Find the maximum and minimum of $xy \cdot \log(x^2 + y^2)$ on $x^2 + y^2 \leq r$.

8.1.65. (1) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivative $D_1 f \equiv 0$, then f only depends on y.

8.1.66. (2) Prove that $(x_1, x_2, \ldots, x_n) \mapsto x_1 + x_2 + \ldots + x_n$ is differentiable. What is its derivative? For a given ε find δ!

8.1.67. (2) Prove that $(x, y) \mapsto x^y$ is continuously differentiable on $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. What is the derivative?

8.1.68. (3) Prove that $f(x, y) = \dfrac{x^3}{x^2 + y^2}$, $f(0, 0) = 0$ has directional derivatives at the origin in all directions. Is there a vector a such that for all v unit vector one has $D_v f(0, 0) = a \cdot v$?

8.1.69. (4) Describe those $f : \mathbb{R}^2 \to \mathbb{R}$ for which $D_1 f \equiv D_2 f$?

8.1.70. (4) Prove that if $f : \mathbb{R}^p \to \mathbb{R}$ is differentiable at a, $f(a) = 0$ and $f'(a) = 0$, then for all bounded $g : \mathbb{R}^p \to \mathbb{R}$, gf is differentiable at a.

8.1.71. (5) Give a function g whose directional derivatives all exist and vanish at the origin, but
(a) g is not differentiable at the origin;
(b) not continuous at the origin;
(c) not bounded in any neighborhood of the origin.

8.1.72. (6) Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ has a second partial derivative $D_{12} f$ which is non-negative. Show that if $a < b$ and $c < d$, then $f(a, c) + f(b, d) \geq f(a, d) + f(b, c)$.

8.1.73. (5) Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ has a second partial derivative $D_{12} f$ and for all $a < b$, $c < d$ we have $f(a, c) + f(b, d) \geq f(a, d) + f(b, c)$. Show that D_{12} is non-negative.

8.1.74. (5) Find the derivative of $\operatorname{tr} : \mathbb{R}^{n \times n} \to \mathbb{R}$, $\operatorname{tr} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \ldots + a_{nn}$.

8.1.75. (2) Find the derivative of the scalar product of n-dimensional vectors when viewed as an $\mathbb{R}^{2n} \to \mathbb{R}$ function.

8.1.76. (1) Prove that $(x, y) \mapsto x/y$ is differentiable $(y \neq 0)$. What is the derivative?

8.1.77. (2) Prove that $(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$ is differentiable. What is the derivative?

8.1.78. (5) True or false? If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and for all lines through a f has a local minimum at a along the line, then f has a local minimum at a.

8.1.79. (5) Let B be a real $q \times r$ matrix. What is the derivative of

$$f(x_1, \ldots, x_{q+r}) = (x_1, \ldots, x_q)M(x_{q+1}, \ldots, x_{q+r})^T?$$

8.1.80. (5) True or false? If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at all points except perhaps at the origin and at the origin it has vanishing directional derivatives in all directions, then f is differentiable at the origin.

8.1.81. (3)

8.1.82. (4) For which values of $\alpha, \beta > 0$ is $|x|^\alpha \cdot |y|^\beta$ twice differentiable at the origin?

8.1.83. (1) Write down the second degree Taylor polynomial of xyz at $(1, 2, 3)$.

8.1.84. (1) Write down the third degree Taylor polynomial of $\sin(x + y)$ at $(0, 0)$.

8.1.85. (3) Find the local extrema of

$$x^2 + xy + y^2 - 3x - 3y + 5; \qquad x^3y^2(2 - x - y).$$

8.1.86. (8) Prove that if $D_{12}f$ and $D_{21}f$ exist in a neighborhood of (a, b) and they are both continuous at (a, b), then $D_{12}f(a, b) = D_{21}f(a, b)$.

8.1.87. (8) Prove that if D_1f, D_2f and $D_{12}f$ exist in a neighborhood of (a, b) and D_{12} is continuous at (a, b), then D_{21} exists and $D_{12}f(a, b) = D_{21}f(a, b)$. (Schwarz)

8.1.88. (3) Find the local extrema of the following functions:

$$x^3 + y^3 - 9xy; \qquad \sin x + \sin y + \sin(x + y)$$

8.1.89. (7) Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and for all x, y we have

$$y^2 \cdot D_1f(x, y) = x^2 \cdot D_2f(x, y).$$

Prove that $f(x, y) = g(x^3 + y^3)$ for some g. Is it necessarily true that the function g is differentiable at 0?

8.1.90. (3) Prove that if $f_1, ..., f_p : \mathbb{R} \to \mathbb{R}$ are twice differentiable and convex, then $g(x_1, \ldots, x_p) = f_1(x_1) + \ldots + f_p(x_p)$ is also convex.

8.1.91. (3) What are the local extrema of $xy + \frac{1}{x} + \frac{1}{y}$?

8.1.92. (5) How many local maximum and minimum places exist for $(1 + e^y)\cos x - ye^y$?

8.1.93. (2) Let $f(x, y) = \psi(x - ay) + \varphi(x + ay)$, where ψ, φ are smooth.

$$\frac{\partial^2 f}{\partial y^2} - a^2 \frac{\partial^2 f}{\partial x^2} = ?$$

8.1.94. (4) For what c is

$$f(x, y) = \begin{cases} \frac{|x|^c y}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

differentiable?

8.1.95. (7) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $D_1 f(x, y) = yD_2 f(x, y)$ for all x, y, then there is a $g : \mathbb{R} \to \mathbb{R}$ differentiable function for which $f(x, y) = g(e^x y)$.

8.1.96. (7) Prove that if $H \subset \mathbb{R}^p$ is convex and open and $f : H \to \mathbb{R}$ is convex, then f is Lipschitz on all compact subsets of H.

8.1.97. (9) Given $F : \mathbb{R}^p \to \mathbb{R}$ twice differentiable convex function we are looking for the minimum of F using the *conjugate gradient method*: start with x_0 and let

$$x_{n+1} = x_n - c(x_n) \cdot \operatorname{grad} f(x_n),$$

where $c(x_n)$ is computed from the first and second derivatives of f at x_n.

(a) What is a good choice for $c(x_n)$?

(b) Prove that the method works for quadratic forms.

8.1.98. (4) Let $H \subset \mathbb{R}^{p+q}$, $a \in \mathbb{R}^p$, $b \in \mathbb{R}^q$, $(a, b) \in \operatorname{int} H$ and $f : H \to \mathbb{R}$ differentiable at (a, b) and assume that near a there is a differentiable function φ to $\mathbb{R}^q$ such that $f(x, \varphi(x)) = 0$. Prove that

$$f'_a(b) \circ \varphi'(a) = -(f^b)'(a).$$

8.1.99. (4) For $|x| < 1$, $|y| < 1$, $|z| < 1$ let $u(x, y, z)$ be the real root of

$$(2 + x)u^3 + (1 + y)u - (3 + z) = 0.$$

Find $u'(0, 0, 0)$.

8.1.100. (4) For $|x_1 - 10| < 1$, $|x_2 - 20| < 1$, $|x_3 - 30| < 1$ let $u = (u_1, u_2)$ be the root of

$$u_1 + u_2 = x_1 + x_2 + x_3 - 10, \quad u_1 u_2 = \frac{x_1 x_2 x_3}{10}$$

closest to $(30, 20)$. Find $u'(10, 20, 30)$.

8.1.101. (4) Given the constraints $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ find the largest possible values of x, $x + y + z$, and $y + z$.

8.1.102. (4) Find the maximum of xyz given the constraints $x + y + z = 5$ and $x^2 + y^2 + z^2 = 9$.

8.1.103. (5) Let A and B be $n \times n$ real symmetric matrices where $\det A \neq 0$.

(a) Prove that if $x \to x^T Bx$ has a local extremum at $x_0 \in \mathbb{R}^n$ given the constraint $x^T Ax = 1$, then x_0 is an eigenvector of $A^{-1}B$.

(b) What is the meaning of the eigenvalue corresponding to the eigenvector x_0?

8.1.104. (6) Given $p_1, \dots, p_n$ in 3-space we are looking for the plane through the origin for which the sum of the squared distances from the points to the plane is minimal. Let v be the normal vector of this plane, where $|v| = 1$.

(a) Show that v is an eigenvector of the matrix $\sum\limits_{i=1}^{n} p_i p_i^T$.

(b) What is the geometric meaning of the eigenvalue corresponding to the eigenvector v?

8.1.105. (4) What is the image of $x^2 + y^2 \leq 1$ under the map $x^2 y^3 \log(x^2 + y^2)$?

8.1.106. (5) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable. Prove that if

$$\langle f'(x, y, z), (x, y, z) \rangle \geq 0$$

holds everywhere, then

$$D_{11} f(0, 0, 0) + D_{22} f(0, 0, 0) + D_{33} f(0, 0, 0) \geq 0.$$

8.1.107. (4) Find the distance of $(5,5)$ from the hyperbola $xy = 4$ using Lagrange multiplicators.

8.1.108. (7) We know that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and

$$y^2 \cdot D_1 f(x,y) + x^3 \cdot D_2 f(x,y) = 0.$$

Prove that $f(\sqrt{2}, \sqrt[3]{3}) = f(0,0)$.

8.1.109. (4) Is the function

$$f(x,y,z) = \begin{cases} \frac{\sin^2 x + \sin^2 y + \sin^2 z}{x^2+y^2+z^2} & (x,y,z) \neq (0,0,0) \\ 1 & x = y = z = 0 \end{cases}$$

differentiable at the origin?

8.2 Vector Valued Functions of Several Variables

8.2.1 Limit and Continuity

8.2.1. (5) $f : \mathbb{R}^p \to \mathbb{R}^q$, $A, B \subset \mathbb{R}^p$, $x \in A \cap B$. Assume that f is continuous at x when restricted to either A or B. Prove that f is continuous at x when restricted to $A \cup B$. Does this remain true for a union of infinitely many sets?

8.2.2. (3) $f : \mathbb{R}^p \to \mathbb{R}^q$, $A, B \subset \mathbb{R}^p$. Assume that f is continuous when restricted to either A or B. Is it true that f is continuous when restricted to $A \cup B$?

8.2.3. (3) Let $f : \mathbb{R}^p \to \mathbb{R}^q$, $A, B \subset \mathbb{R}^p$ be closed. f is continuous when restricted to either A or B. Is it true that f is continuous when restricted to $A \cup B$?

8.2.4. (10)

8.2.2 Differentiation

8.2.5. (3) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x,y) \mapsto (e^x, x^2+y^2, \sin x)$; $g : \mathbb{R}^3 \to \mathbb{R}$, $(X,Y,Z) \mapsto XY$. $(g \circ f)' = ?$

8.2.6. (1) Find the Jacobi-matrix of the following functions

$$f(x,y) = \big(x+y, xy, \cos(x+y)\big); \quad g(x,y) = \big(e^{x+y}, xy\big); \quad h = f \circ g.$$

8.2.7. (2) Prove that vectorial product viewed as a $\mathbb{R}^6 \to \mathbb{R}^3$ function is differentiable. What is its derivative?

8.2.8. (4) What is the Jacobi matrix of the local inverse of $f(x,y) = (x^2 - y^2, 2xy)$?

8.2.9. (5) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Show that

$$||A^{-1}|| = \frac{1}{\min\{Ax | x \in S_0^{n-1}(1)\}}.$$

8.2.10. (5) Find an $A : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation for which

$$\sqrt{\sum_{i,j} a_{i,j}^2} > ||A||.$$

Show that $\geq$ is always true.

8.2.11. (8) Prove that

$$\max_{1\leq j\leq p} \sqrt{\sum_{i=1}^{q} a_{ij}^2} \leq \left\| \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{q1} & \dots & a_{qp} \end{pmatrix} \right\| \leq \sqrt{\sum_{i=1}^{q}\sum_{j=1}^{p} a_{ij}^2}.$$

Give an example when equality does not hold.

8.2.12. (2) Find the Jacobi-matrix of the following functions:

$$f(x,y) = \big(\sin x, \cos y\big); \quad g(x,y) = \big(\log x, x^2 + y^2\big); \quad h = f \circ g.$$

8.2.13. (4) Let $f : \mathbb{R}^p \to \mathbb{R}^q$ be differentiable at the points of the interval $[a,b] \subset \mathbb{R}^p$. Prove that

$$|f(b) - f(a)| \leq |b - a| \cdot \sup_{c\in[a,b]} ||f'(c)||.$$

8.2.14. (7) Prove that for all $A \in \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^p)$ $||A|| \geq |\det A|^{1/p}$.

8.2.15. (5)

(a) Prove that all linear maps $\mathbb{R}^p \to \mathbb{R}^q$ are Lipschitz.

(b) Prove that if $A \in \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^p)$ is invertible, then $\exists c > 0\, \forall x \in \mathbb{R}^p\, |A(x)| \geq c|x|$.

Chapter 9

Jordan Measure and Riemann Integral in Higher Dimensions

9.0.1. (2) Prove that for all $0 \le a \le b$ there exists a bounded set $H \subset \mathbb{R}^p$ for which $b(H) = a$ and $k(H) = b$.

9.0.2. (3) Let $H \subset \mathbb{R}^p$ be a bounded set. Determine whether the following statements are true or false.

(a) If $k(H) = 0$, then $H \in \mathcal{J}$.
(b) If $H \in \mathcal{J}$, then $\partial H \in \mathcal{J}$.
(c) If $\partial H \in \mathcal{J}$, then $H \in \mathcal{J}$.
(d) If $H \in \mathcal{J}$, then $\operatorname{int} H \in \mathcal{J}$.
(e) If $H \in \mathcal{J}$, then $\operatorname{cl} H \in \mathcal{J}$.
(f) If $\operatorname{int} H \in \mathcal{J}$ and $\operatorname{cl} H \in \mathcal{J}$, then $H \in \mathcal{J}$.

9.0.3. (5) Let $A, B \subset \mathbb{R}^p$ be disjoint bounded sets. Order the following numbers

$$k(A \cup B); \qquad b(A \cup B); \qquad k(A) + k(B); \qquad b(A) + b(B);$$

$$k(A) + b(B); \qquad b(A) + k(B).$$

9.0.4. (5) Let $f : (0,1) \to \mathbb{R}$, $f(x) = x \sin \log x$. Is this a function of bounded variation? Is it absolutely continuous?

9.0.5. (4) Determine whether the following statements are true or false. Here f is a function from $[a,b]$ to $\mathbb{R}$.

(a) If f is monotonic, then f is of bounded variation.

(b) If f is continuous, then f is of bounded variation.

(c) If f is continuous and of bounded variation, then f is Lipschitz.

(d) If f is of bounded variation, then the interval $[a,b]$ can be written as the union of countable many subintervals on each of which f is monotonic.

(e) If the $\int_a^b \mathrm{d}f$ Stieltjes integral exists, then f is absolutely continuous.

(f) If f is absolutely continuous, then f is Riemann-integrable.

9.0.6. (5) Let $H \subset \mathbb{R}^p$ be a bounded set. Are the following statements true or false?

(a) If $\operatorname{cl} H \in \mathcal{J}$, then $H \in \mathcal{J}$.

(b) If H is closed and $H \in \mathcal{J}$, then $\operatorname{int} H \in \mathcal{J}$.

(c) If H is open and $H \in \mathcal{J}$, then $\operatorname{cl} H \in \mathcal{J}$.

(d) If $k(\operatorname{int} H) = b(\operatorname{cl} H)$, then $H \in \mathcal{J}$.

(e) $\partial H \in \mathcal{J}$.

9.0.7. (4) Let $A \subset \mathbb{R}^p$, $B \subset \mathbb{R}^q$ be bounded sets. True or false?

(a) $k^{(p+q)}(A \times B) = k^{(p)}(A) \cdot k^{(q)}(B)$.

(b) $b^{(p+q)}(A \times B) = b^{(p)}(A) \cdot b^{(q)}(B)$.

(c) If A and B are measurable, then $A \times B$ is also measurable and $t^{(p+q)}(A \times B) = t^{(p)}(A) \cdot t^{(q)}(B)$.

9.0.8. (6) Let $A_1, \dots, A_n$ be measurable sets in the unit cube whose measures add up to more than k. Show that there is a point which is contained in at least k of these sets.

9.0.9. (5) Prove that if $A \subset B \subset \mathbb{R}^p$ and B is Jordan-measurable, then

$$t(B) = k(A) + b(B \setminus A).$$

9.0.10. (5) Show that a bounded set $A \subset \mathbb{R}^p$ is measurable if and only if

$$k(B) = k(B \cap A) + k(B \setminus A)$$

for any set $B \subset \mathbb{R}^p$.

9.0.11. (5) Let $A \subset [a,b]$ be Jordan-measurable. Connect the points of A to an arbitrary (but fixed) point of the plane. Show that the union of these line segments is Jordan-measurable in the plane. What is its “area”?

9.0.12. (4) Is it true that if $A \subset \mathbb{R}$ is measurable, then

$$\{(x, y) : \sqrt{x^2 + y^2} \in A\} \subset \mathbb{R}^2$$

is measurable?

9.0.13. (7) Prove that if $B_1, B_2, \ldots \subset \mathbb{R}^p$ are pairwise disjoint open balls, then

$$b\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} b(B_i).$$

9.0.14. (7) Show that for any $0 \leq c \leq d < \infty$ there exists a bounded, closed set with interior measure c, and exterior measure d.

9.0.15. (6) Prove that if $m : \mathcal{J} \to \mathbb{R}$ is non-negative, additive, translation-invariant and normed, then $m = t$.

9.0.16. (5) Prove that if $A, B \subset \mathbb{R}^p$ and $\operatorname{cl} A \cap \operatorname{cl} B$ is of measure zero, then $k(A \cup B) = k(A) + k(B)$.

9.0.17. (6) Prove that a bounded set $A \subset \mathbb{R}^p$ is measurable if and only if

$$b(B) = b(B \cap A) + b(B \setminus A)$$

for any set $B \subset \mathbb{R}^p$.

9.0.18. (5) Let $A \subset \mathbb{R}^p$ be Jordan-measurable. Is it true that the set $\bigcup_{a \in A} [0, a]$ is measurable?

9.0.19. (6) For any $\varepsilon > 0$ divide the n-dimensinal unit cube into an open and closed part in such a way that the inner Jordan measure of each is less than ε.

9.0.20. (10) For any $H \subset \mathbb{R}^p$ bounded set let $B(H)$ be (a) largest open ball in H if H has no interior, then let $B(H) = \emptyset$. Starting from an $A_0 \subset \mathbb{R}^p$ Jordan-measurable set let $A_1 = A$ and $A_{n+1} = A_n \setminus B(A_n)$. Prove that $\lim b(A_n) = 0$.

9.0.21. (9) Is there a Peano-curve that is differentiable? (I.e. is there a surjective differentiable map $[0, 1] \to \mathbb{R}^2$?)

9.0.22. (8) Let $f : [0,1] \to \mathbb{R}^2$ be a simple closed curve. Does it follow that its image has measure 0?

9.0.23. (3) What is the moment of inertia for a cylinder of mass m, radius r, and height $2h$ about an axis that goes through its center but is orthogonal to its axis of symmetry?

9.0.24. (2) Interchange the order of integration.

$$\int_0^1 \int_x^{2x} f(x,y)\, \mathrm{d}y\, \mathrm{d}x; \qquad \int_{-1}^1 \int_{|x|}^{x^2+x+1} f(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

9.0.25. (3)

$$\int_0^1 \int_0^x y^2 e^x\, \mathrm{d}y\, \mathrm{d}x = ?$$

9.0.26. (4) The vertices of a triangle are $A = (a,0)$, $B = (b,0)$ and $C = (0,m)$. For $(x,y) \in [0,1]^2$ let

$$f(x,y) = (1-x)(1-y)\cdot A + x(1-y)\cdot B + y \cdot C.$$

Use this map and the theorem on measure transformation to determine the area of the triangle.

9.0.27. (3) Calculate the area of the set, defined with polar coordinates, by $\beta - 90^\circ \le \varphi \le 90^\circ - \gamma$, $0 \le r \le \dfrac{m}{\cos\varphi}$.

9.0.28. (3)

$$\int_{\pi^2 \le x^2+y^2 \le 4\pi^2} \sin(x^2+y^2)\, \mathrm{d}x\, \mathrm{d}y = ?$$

9.0.29. (7) Prove that if A is measurable with positive measure and f is integrable on A, then there is at least one point where f is continuous.

9.0.30. (5) Let f be bounded and non-negative on the measurable set A. Prove that $\int_A f = 0$ implies that $k(\{x \in A : f(x) \ge a\}) = 0$ for all $a > 0$. Is the converse true?

9.0.31. (10) We need a simulated random sequence of normal distribution, i.e. with density $\varrho(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Given a random-number generator that gives random numbers with uniform distribution in $[0,1]$ how can one generate such a sequence. (Hint: use two sequences.)

9.0.32. (8) For all continuous functions $f : \mathbb{R} \to \mathbb{R}$ let $I_0 f = f$ and for $a \geq 0$ let $I_a f$ be the function for which

$$(I_a f)(x) = \int_0^x f(t)\frac{(x-y)^{a-1}}{\Gamma(a)}\,\mathrm{d}x.$$

Prove that (a) $(I_1 f)(x) = \int_0^x f$; (b) $I_{a+b} = I_a I_b$.

9.0.33. (5) Prove Steiner's theorem: if a rigid body has mass m and its moment of inertia about an axis l through its center of mass is I, then the moment of inertia about an axis parallel to l and of distance r is $I + mr^2$.

9.0.34. (4)

$$\int_0^1 \left(\int_{\sqrt{y}}^1 \sqrt{1+x^3}\,dx\right) dy = ? \qquad \int_0^1 \left(\int_{y^{2/3}}^1 y\cos x^2\ dx\right) dy = ?$$

9.0.35. (3) Calculate the volume of $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1,\ |z| \leq e^{\sqrt{x^2+y^2}}\}$.

9.0.36. (7) Is it true that if $f : [0,1] \times [0,1] \to \mathbb{R}$ is monotonic on every horizontal and vertical segments, then it is integrable?

9.0.37. (7) Prove that if $f > 0$ on $A \subset \mathbb{R}^n$ with positive Jordan measure, then $\overline{\int}_A f\,dx > 0$.

9.0.38. (10) Let $a \in \mathbb{R}$. $\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cos(ax)\,\mathrm{d}x = ?$

9.0.39. (6) Prove that a bounded set $K \subset \mathbb{R}^n$ is Jordan-measurable if and only if it cuts all bounded open sets "properly" i.e. for all bounded open set $X \subset \mathbb{R}^n$ one has $b(X \cap K) + b(X \setminus K) = b(X)$.

9.0.40. (6) Prove that a bounded set $K \subset \mathbb{R}^n$ is Jordan-measurable if and only if it cuts all bounded closed sets "properly" i.e. for all bounded closed set $X \subset \mathbb{R}^n$ one has $k(X \cap K) + k(X \setminus K) = b(X)$.

9.0.41. (4) Give a function $\varphi : [0,2] \to \mathbb{R}$ such that for any continuous function $f : [0,1] \to \mathbb{R}$

$$\int_0^1 \int_0^1 f(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 f\varphi.$$

9.0.42. (4)

$$\int_0^{\pi/2} \left(\int_x^{\pi/2} \frac{\sin y}{y}\,\mathrm{d}y \right) \mathrm{d}x = ?$$

9.0.43. (3) What is the moment of inertia of a cone about its axis of rotation if it has homogeneous mass distribution with mass m, its height is h and its base disc has radius r?

9.0.44. (8) Prove that if $F_1 \supset F_2 \supset \ldots$ are bounded, closed sets and $\bigcap_{n=1}^{\infty} F_n$ is of measure zero, then $k(F_n) \to 0$.

9.0.45. (9) Let $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,\mathrm{d}x$ and $B(s,u) = \int_0^1 x^{s-1}(1-x)^{u-1}\,\mathrm{d}x$ be Euler's Gamma and Beta functions. Show that

$$B(s,u) = \frac{\Gamma(s)\Gamma(u)}{\Gamma(s+u)}.$$

9.0.46. (7) Express the volume of the n-dimensional unit ball using Euler's Γ function. What is the volume of the "half-dimensional" unit ball?

9.0.47. (4) Prove that $\sum_{n=1}^{\infty} e^{-n^2 x}$ is infinitely differentiable on $(0,\infty)$.

9.0.48. (4)

$$\int_0^1 \sqrt{x} \left(\int_{x^{3/4}}^1 e^{y^3}\,\mathrm{d}y \right) \mathrm{d}x = ?$$

9.0.49. (7) Prove that for $s > 0$ $\Gamma(s) \cdot \Gamma''(s) > \left|\Gamma'(s)\right|^2$.

9.0.50. (7) Formulate and prove the Dirichlet and Abel criterions for improper integrals.

9.0.51. (6) Formulate a Weierstrass type criterion for improper Stieltjes integrals.

9.0.52. (7) Is $f(t) = \int_1^t \int_1^t e^{xyt}\ \mathrm{d}x\ \mathrm{d}y$ $(t > 1)$ differentiable? What is its derivative?

9.0.53. (7) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous, and
$G(r) = \int_{x^2+y^2 \le r^2} f(x, y, r)\ \mathrm{d}x\ \mathrm{d}y$ $(r > 0)$.

(a) Show that G is continuous.

(b1) Show that if f continuously differentiable, then G is also continuously differentiable. What is G'?

(b2) Can the condition of continuous differentiablity be weakened?

9.0.54. (8) Prove that Euler's Beta function is strictly convex.

9.0.55. (7) Is $f(t) = \int_1^t e^{x^2 t}\ \mathrm{d}x$ differentiable? What is its derivative?

9.0.56. (7) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $G(x) = \int_{-x}^{x^2} f(x, y)\ \mathrm{d}y$.

(a) Prove that G is continuous.

(b1) Show that if f is continuously differentiable, then G is also continuously differentiable. What is G'?

(b2) Can the condition of continuously differentiability weakened?

9.0.57. (5) Show that Euler's Beta function is infinitely differentiable and express its derivative as an integral.

9.0.58. (10) According to Tauber's theorem if $\lim_{r \to 1-0} \sum_{n=0}^{\infty} a_n r^n = C$ exists and finite and moreover $n a_n \to 0$, then $\sum_{n=0}^{\infty} a_n = C$.

(a) Formulate a Tauberian theorem for parametric integrals.

(b) Prove the Tauberian theorem for parametric integrals you formulated.

9.0.59. (10) For $x \in \mathbb{R}$ let $I(x) = \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \cos(xt)\ \mathrm{d}t$.

(a) Prove that $I(x) \cdot I(y) = I\left(\sqrt{x^2 + y^2}\right)$.

(b) Describe the behavior of I near 0.

(c) $I(x) =?$

9.0.60. (9) Let B be Euler's Beta function. Prove that $\log B$ is convex.

Chapter 10

The Integral Theorems of Vector Calculus

10.1 The Line Integral

10.1.1. (3) Let $\gamma : [1,2] \to \mathbb{R}^3$, $\gamma(t) = (\log t, 2t, t^2)$.
(a) Determine the length of γ.
(b) Determine the line integral of the vector field $f(x, y, z) = (x, y, z)$ along the curve γ.

10.1.2. (3) Let C be the geometric curve $\{(x, y) | \, |x| + |y| = a\}$. $\int_C xy \, \mathrm{d}s =?$

10.1.3. (3) Let $\gamma : [0, 2] \to \mathbb{R}^2$, $(t) \mapsto (t, t^2)$. Compute the $\int_\gamma (-y, x) \, \mathrm{d}g$ line integral where g is the identity function.

10.1.4. (3) Let γ be the semicircle which is the right part of the circle centered at 0 with radius a (i.e. those points satisfying $x \geq 0$). $\int_\gamma x \, \mathrm{d}y =?$

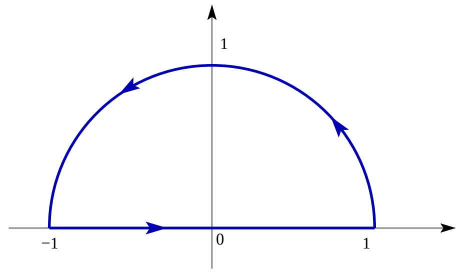
10.1.5. (3) Let γ be the semicircle which is the upper part of the circle centered at 0 with radius a (i.e those points satsifying $y \geq 0$). $\int_\gamma x^2 \, \mathrm{d}s =?$

10.1.6. (4)

$$a) \quad \int_0^2 \sin x \, \mathrm{d}\{x\} =? \qquad\qquad b) \quad \int_\gamma x^2 \, \mathrm{d}(y^2) =?$$

where γ is the triangle with vertices $(0, 0), (2, 0), (0, 1)$.

10.1.7. (4) Calculate the line integral $\int xy \ \mathrm{d}y$ on the curve in the figure.



10.1.8. (3) Determine the line integral of the vector field $\left(\dfrac{x}{1+y}, \dfrac{y}{2+x}\right)$ along the parabola $y = x^2$ segment between the points $(-1, 1)$ and $(1, 1)$.

10.1.9. (4) Consider a map $g : [a, b] \to \mathbb{R}$ as a one-dimensional curve. When is it rectifiable? What is its length?

10.1.10. (4) Let $g : [0, 1] \to \mathbb{R}^2$ be a simple closed and rectifiable curve. Prove that

$$\int_g x^2 \ \mathrm{d}x = \int_g e^{-\cos y^2} \ \mathrm{d}y = 0.$$

10.1.11. (4) Let $* : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^r$ be bilinear, $f : \mathbb{R}^q \to \mathbb{R}^p$ continuous and $g : [a, b] \to \mathbb{R}^q$ a continuous curve. Show that

(a) if g is rectifiable, then $\int_g f(\mathbf{x}) * \mathrm{d}\mathbf{x}$ exists;

(b) if g is continuously differentiable, then $\int_g f(\mathbf{x}) * \mathrm{d}\mathbf{x} = \int_a^b f(g(t)) * g'(t)\,\mathrm{d}t$.

10.2 Newton-Leibniz Formula

10.2.1. (3) Let $g(t) = (t, t^2)$ $(t \in [0, 1])$. Calculate the line integrals:

$$\int_g \cos x \ \mathrm{d}y \qquad \int_g \langle (e^x \cos x, e^x \sin y), \mathrm{d}\mathbf{x} \rangle$$

10.2.2. (3) Let $g(t) = (1, t, t^2)$ $(t \in [0, 1])$ and $f(x, y, z) = (yz, xz, xy)$. Calculate the following line integrals:

$$\int_g f_1 \ \mathrm{d}x_2 \qquad \int_g \langle f, \ \mathrm{d}\mathbf{x} \rangle \qquad \int_g f \times \mathrm{d}\mathbf{x}$$

Which of these integrals can be computed immediately from the fundamental theorem of calculus for line integrals?

10.2.3. (4) For what functions $f : \mathbb{R}^2 \to \mathbb{R}$ will the following statement be true? If g is a simple, closed rectifiable curve in $\mathbb{R}^2$, then

$$\int_g x^2 y^3 \, \mathrm{d}y = \int_g f(x, y) \, \mathrm{d}x.$$

Answer→

10.2.4. (5) What differentiable $f : \mathbb{R}^2 \to \mathbb{R}$ functions satisfy the following statement? If g is a simple closed rectifiable curve in $\mathbb{R}^2$, then

$$\int_g e^x \cos y \, \mathrm{d}x = \int_g f(x, y) \, \mathrm{d}y.$$

10.2.5. (5) Give a continuous vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose line integral vanishes on every closed rectifiable curve, but which is not everywhere differentiable.

10.2.6. (7) Show that if the line integral of a continuous $f : \mathbb{R}^2 \to \mathbb{R}^2$ vanishes on any rectangles whose sides are parallel to the axes, then f is a gradient field.

Related problem: 10.3.5

10.3 Existence of the Primitive Function

10.3.1. (2) Which sets are simply connected?

$$\mathbb{R}^2 \setminus (\mathbb{Z} \times \mathbb{Z}) \qquad \mathbb{R}^3 \setminus (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \qquad \mathbb{R}^3 \setminus \{(\cos t, \sin t, 0) : t \in \mathbb{R}\}$$
$$\mathbb{R}^4 \setminus \{(\cos t, \sin t, 0, 0) : t \in \mathbb{R}\}$$

10.3.2. (3) Let $G \subset \mathbb{R}^p$ be open and connected. Show that the scalar potentials of a vector field $G \to \mathbb{R}^p$ can differ only in constants.

10.3.3. (5) Which of the following is simply connected?

$$\mathbb{R}^2 \setminus \{(0,0)\} \quad \mathbb{R}^3 \setminus \{(0,0,0)\} \quad \mathbb{R}^3 \setminus \{t,0,0) : t \in \mathbb{R}\} \quad \mathbb{R}^4 \setminus \{t,0,0,0) : t \in \mathbb{R}\}$$

10.3.4. (10) Let $G \subset \mathbb{R}^p$ be open, let $f : G \to \mathbb{R}^p$ be differentiable and irrotational and let $g, h : [0,1] \to G$ be continuously differentiable curves with the same initial and end points. (I.e. $g(0) = h(0)$ and $g(1) = h(1)$.) Assume that g and h are homotopic, $\exists \varphi : [0,1]^2 \to \mathbb{R}^p$ continuous such that $\varphi(t,0) = g(t)$, $\varphi(t,1) = h(t)$, and $\varphi(0,u) = g(0) = h(0)$, $\varphi(1,u) = g(1) = h(1)$ for all $u \in [0,1]$.

(a) Show from Goursat's lemma that $\int_g \langle f, \mathrm{d}x \rangle = \int_h \langle f, \mathrm{d}x \rangle$.

(b) Assume in addition that φ is continuously differentiable $I(u) = \int_{\varphi(\cdot,u)} \langle f, \mathrm{d}x \rangle$. Prove directly that $I' = 0$.

10.3.5. (6) Redo the proof of Goursat's lemma for rectangles.

10.3.6. (5) Let $H = \mathbb{R}^3 \setminus \{(x,y,0) : \ x^2 + y^2 = 1\}$. Give a differentiable irrotational vector field $H \to \mathbb{R}^3$ which is not a gradient field.

10.3.7. (5) Which of the following vector fields are gradient fields? For those that are not, give a closed curve on which the line integral of the field does not vanish.

$$(x,y) \qquad (y,x) \qquad \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \qquad \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

10.3.8. (5) Let $G = \mathbb{R}^3 \setminus \{(x,x,x) : x \in \mathbb{R}\}$. Find a differentiable vector field $X : G \to \mathbb{R}^3$ that is irrotational ($\operatorname{curl} X = \mathbf{0}$) but is not a gradient field.

10.3.9. (4) Which of the following vector fields are gradient fields? For those that are not, give a closed curve on which the line integral of the field does not vanish.

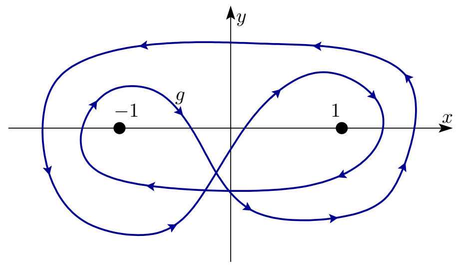
$$(\cosh y; x \sinh y) \qquad (\cosh x; y \sinh x) \qquad \left(\frac{x}{x^2+y^2}; \frac{y}{x^2+y^2};\right)$$

10.3.10. (3) The electric field of a homogeneously charged line is orthogonal to the line and its strength at distance d from the line is $2k\rho/d$. Determine the electric potential difference (voltage) between two points.

10.3.11. (9) Is $H = \mathbb{R}^3 \setminus \{(\cos t, \sin t, e^t) : t \in \mathbb{R}\}$ simply connected?

Answer→

10.3.12. (10) Let $G = \mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$, and g be the curve shown in the figure.



(a) Show that the line integral of any differentiable irrotational vector field $f : G \to \mathbb{R}^2$ along g is zero.
(b) Is g homotopic to a point in G?
(c) Is g homologous to 0 in G?

10.3.13. (8) Let $G \subset \mathbb{R}^2$ be open and let $\varphi_u(t)$ $[0,1]^2 \to G$ be continuously differentiable family of curves. Show that for a continuously differentiable $f : G \to \mathbb{R}^2$ irrotational vector field the $I(u) = \int_{\varphi_u} \langle f, \mathrm{d}x \rangle$ parametric line integral satisfies $I'(u) = 0$.

10.4 Integral Theorems

10.4.1. (1) Check the statement of Green's theorem for $[0,1] \times [0,1]$ and the function $f(x,y) = xy$.

10.4.2. (5) What are the one-dimensional versions of gradient, divergence, rotation and the divergence and Stokes theorems?

10.4.3. (2) For a fixed $a \in \mathbb{R}^3$ let $f(x) = a \times x$ and $g(x) = x \times a$ $(x \in \mathbb{R}^3)$.

$$\operatorname{div} f = ? \qquad \operatorname{div} g = ? \qquad \operatorname{rot} f = ? \qquad \operatorname{rot} g = ?$$

10.4.4. (3) From the 9 possible compositions of div, rot, grad which ones are meaningful? Which ones produce zero?

10.4.5. (5) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Show that

$$\operatorname{rot}\operatorname{rot} f = \operatorname{grad}\operatorname{div} f - \begin{pmatrix} \operatorname{div}\operatorname{grad} f_1 \\ \operatorname{div}\operatorname{grad} f_2 \\ \operatorname{div}\operatorname{grad} f_3 \end{pmatrix}.$$

10.4.6. (8) Let g be the polygonal boundary of the convex set $F \subset \mathbb{R}^3$. Show that

$$\vec{A} = \frac{1}{2} \int_g \mathbf{x} \times \mathrm{d}\mathbf{x}$$

is the right-handed area vector.

10.4.7. (3) Let $P = \{(u, v) \in [0, 1]^2 : u^2 + v^2 \leq 1\}$, $g(u, v) = (u, v, u^2 + v^2)$, $F = g(P)$ and $f(x, y, z) = (x, y, z)$. Rewrite the following surface integrals as Riemann integrals of one or more variables.

$$\int_F \overrightarrow{\mathrm{d}S}; \quad \int_F |\,\mathrm{d}S|; \quad \int_F \left\langle f,\ \overrightarrow{\mathrm{d}S} \right\rangle; \quad \int_F f \times \overrightarrow{\mathrm{d}S}.$$

10.4.8. (4) Compute the surface area of a sphere of radius r using the divergence theorem for the vector field $f(x, y, z) = (x, y, z)$.

10.4.9. (9) Let F be a continuously differentiable parametric surface in $\mathbb{R}^3$ that is bounded by the closed simple and rectifiable curve g in such a way that the preimage of g in the parametrization is positively oriented. Show that if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is continuously differentiable, then

$$\int_F \left\langle \operatorname{rot} f,\ \overrightarrow{\mathrm{d}S} \right\rangle = \int_g \langle f,\ \mathrm{d}x \rangle .$$

10.4.10. (4) Let $B = \left\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\right\}$ and $f(x, y, z) = (yz, x - z, z - y)$.

$$\int_{\partial B} \left\langle f,\ \overrightarrow{\mathrm{d}S} \right\rangle = ?$$

10.4.11. (4) Let $B = \left\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\right\}$ and $f(x, y, z) = (yz, x - z, z - y)$.

$$\int_{\partial B} f \times \overrightarrow{\mathrm{d}S} = ?$$

10.4.12. (7) Let $G \subset \mathbb{R}^2$ be simply connected open and let $g : [0,1] \to G$ be a simple closed rectifiable curve with positive orientation. Let also $A \subset G$ be the bounded component of $\mathbb{R}^2 \setminus g$ and $f : G \to \mathbb{R}^3$ continuously differentiable. Show that

$$\int_A (D_x f \times D_y f)\ \mathrm{d}x\ \mathrm{d}y = \frac{1}{2} \int_{f \circ g} \mathbf{x} \times \mathrm{d}\mathbf{x}.$$

10.4.13. (5) Let $f_1(x, y, z) = xyz$ and $f_2(x, y, z) = x^2 + y^2 + z^2$. Construct a function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ so that the surface integral of vector field (f_1, f_2, f_3) along any closed sphere is the volume of the enclosed ball.

10.4.14. (8) (a) Let $G \subset \mathbb{R}^3$ and $\varphi_t(u, v) : [0,1]^3 \to G$ be a family of continuously differentiable parametric surfaces for which $\varphi_t(u, v)$ is independent of t for any boundary point (u, v) of the unit square. Let also $F : G \to \mathbb{R}^3$ be continuously differentiable and irrotational. Show that the integral

$$I(t) = \int_0^1 \int_0^1 \langle D_x\varphi_t(x, y) \times D_y\varphi_t(x, y), F(\varphi_t(x, y)) \rangle\ \mathrm{d}x\ \mathrm{d}y$$

does not depend on t.

(b) Let $G = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Give an irrotational $H \to \mathbb{R}^3$ vector field whose surface integral along the unit sphere does not vanish.

(c) Show that G is not diffeomorphic to $\mathbb{R}^3$.

Chapter 11

Measure Theory

11.1 Set Algebras

11.1.1. (3) Let $\mathcal{A}$ and $\mathcal{B}$ be σ-rings. Describe the σ-ring generated by $\mathcal{A} \cup \mathcal{B}$.

11.1.2. (7) What is the smallest possible cardinality of an infinite σ-ring?

Answer→

11.1.3. (5) Let $\mathcal{T}$ be the collection of the sets $[a, b) \times [c, d)$.

(a) Show that $\mathcal{T}$ is a semi-ring.

(b) What ring does $\mathcal{T}$ generate?

(c) Show that $f : \mathcal{T} \to \mathbb{R}$ is additive if and only if there is $g : \mathbb{R}^2 \to \mathbb{R}$ for which $f\big([a, b) \times [c, d)\big) = g(b, d) - g(a, d) - g(b, c) + g(a, b)$.

11.1.4. (3) (a) What ring do the half-lines $[a, \infty)$ generate?

(b) What σ-ring do the half-lines $[a, \infty)$ generate?

(c) What is the smallest cardinality of a generating set of the σ-ring of Borel sets?

11.1.5. (5) Show that all open sets are F_σ, and all closed sets are G_δ.

11.1.6. (7) Prove that if $f : \mathbb{R} \to \mathbb{R}$, then the set of points of continuity is Borel, and give as small as possible of Borel class (e.g. $G_{\delta\sigma\delta\sigma\delta\sigma\delta\sigma}$), to which it still belongs.

Solution→

11.1.7. (6) Prove that sets with property F_σ, respectively G_δ, are closed to finite union and intersection.

11.1.8. (5) Show that $F_{\sigma\delta\sigma\delta}(\mathbb{R}^n) \subset G_{\delta\sigma\delta\sigma\delta}(\mathbb{R}^n)$.

11.1.9. (7) Let $f_n : [a, b] \to \mathbb{R}$ be continuous for all n. Prove that $\{x : f_n(x)\,\text{convergent}\}$ is a Borel set, and give a Borel class as small as possible to which it still belongs.

11.2 Measures and Outer Measures

11.2.1. (8) For any $\varepsilon > 0$ give $G \subset \mathbb{R}$ which is open and dense and for which $\overline{\lambda}(G) < \varepsilon$.

11.2.2. (8) Construct a Borel set $H \subset \mathbb{R}$ for which $\lambda((a, b) \cap H) > 0$ and $\lambda((a, b) \setminus H) > 0$ for any $a < b$.

11.2.3. (5) Let μ be a translation-invariant measure on the Borel sets of $\mathbb{R}$, for which $\mu\big([0, 1]\big) < \infty$. Show that μ is the Lebesgue measure up to a constant multiple.

11.2.4. (5) Show that if $H \subset \mathbb{R}$ satisfies $\overline{\lambda}((a, b) \cap H) < \dfrac{99}{100}(b - a)$ for all $a < b$, then H is a null-set.

11.2.5. (9) Can one find continuum many Lebesgue measurable sets in $[0, 1]$ all of measure $1/2$ such that for any two the intersection has measure $1/4$?

11.2.6. (4) Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and for all $a \le b$ let $\mu\big([a, b]\big) = f(b + 0) - f(a - 0)$. What measure does this generate?

11.2.7. (5) Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and μ_f the Lebesgue–Stieltjes measure generated by f. Show that for any Borel set H there are F_σ $B \subset H$ and G_δ $K \supset H$ sets for which $\mu_f(B) = \mu_f(K) = \mu_f(H)$.

11.2.8. (8) (a) Show that if $A \subset \mathbb{R}^p$ is measurable and $\lambda(A) > 0$, then $A - A$ contains a ball centered at the origin (Steinhaus).

(b) Show that if $A, B \subset \mathbb{R}^p$ are measurable with positive measure, then $A + B$ has a non-empty interior.

(c) Show that if $A \subset \mathbb{R}^p$ measurable with positive measure and $B \subset \mathbb{R}^p$ has positive outer measure, then $A + B$ has a non-empty interior.

11.3 Measurable Functions. Integral

11.3.1. (2) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotonic, then it is Borel-measurable.

11.3.2. (2) Prove that the composition of Borel-measurable functions is Borel-measurable.

11.3.3. (4) Show that if $f : [a, b] \to \mathbb{R}$ is Lebesgue-measurable, then there is $g : [a, b] \to \mathbb{R}$ Borel-measurable such that $f = g$ a.e.

11.3.4. (9) Construct a function $f : [0, 1] \to \mathbb{R}$ whose restriction to any set with full measure is not continuous.

11.3.5. (2) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel-measurable, and $g : M \to \mathbb{R}$ measurable for some (M, μ) measure space. Prove that $f \circ g$ is μ-measurable.

11.3.6. (2) True or false? If $f : [a, b] \to \mathbb{R}$ is Riemann-integrable, then it is Borel-measurable.

11.3.7. (2) Let $A \subset \mathbb{R}$ be Lebesgue-measurable and $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. Show that $\int_{\mathbb{R}} \chi_A \,\mathrm{d}\lambda = \lambda(A)$.

11.3.8. (5) Show that if $f > 0$ on a μ-measurable A such that $\mu(A) > 0$, then $\int_A f \,\mathrm{d}\mu > 0$.

11.3.9. (7) True or false? If $f[a, b] \to \mathbb{R}$ is bounded and Lebesgue-integrable, then there is a $g : [a, b] \to \mathbb{R}$ that is Riemann-integrable and for which $f = g$ a.e.

11.3.10. (5) Is there any measurable function $f : \mathbb{R} \to [0, \infty)$, whose integral over any interval is $+\infty$?

11.4 Integrating Sequences and Series of Functions

11.4.1. (8) True or false? If $f_n : \mathbb{R} \to \mathbb{R}$ are Lebesgue-measurable, then they have a subsequence that converges a.e.

11.4.2. (4) Apply Lebesgue's monotone convergence theorem to calculate

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,\mathrm{d}x.$$

11.4.3. (4) True or false? If $f_1 \geq f_2 \geq \dots$ are non-negative and Lebesgue-measurable, then

$$\lim \int f_n \,\mathrm{d}\lambda = \int (\lim f_n)\,\mathrm{d}\lambda.$$

11.4.4. (4) Let $A = \{1,2\}$, and let $\mu : A \to \mathbb{R}$ be the counting measure. State and explain Fatou's lemma in this situation.

11.4.5. (5) Give a sequence $f_n : [0,1] \to \mathbb{R}$ that converges pointwise, for which $\lim \int_0^1 f_n$ exists but $\lim \int_0^1 f_n \neq \int_0^1 \lim f_n$.

11.4.6. (4) Derive the monotone convergence theorem from Fatou's lemma.

11.4.7. (3) State the dominated convergence theorem for series.

11.4.8. (5) True or false? If f_n is non-negative and μ-measurable on a μ-measurable set A and $\int_A f_n d\mu < 1/n$, then $f_n \to 0$ μ-a.e.

11.4.9. (5) Show using the Borel–Cantelli lemma that if f_n is non-negative and μ-measurable on a μ-measurable set A and $\int_A f_n d\mu < 1/n^2$, then $f_n \to 0$ μ-a.e.

11.4.10. (4) Show using the Beppo Levi's theorem that if f_n is non-negative and μ-measurable on a μ-measurable set A and $\int_A f_n d\mu < 1/n^2$, then $f_n \to 0$ μ-a.e.

11.4.11. (8) Show without Lebesgue theory that if $f_n : [0,1] \to [0,1]$ is continuous for all n and $f_n(x) \to 0$ for all $x \in [0,1]$, then $\int_0^1 f_n(x)\,\mathrm{d}x \to 0$!

11.5 Fubini Theorem

11.5.1. (6) Assume the continuum hypothesis and let $\prec$ be a well-ordering of $[0,1]$ of type ω_1. Let

$$A = \{(x,y) \in [0,1]^2 : x \prec y\}.$$

(a) Show that the horizontal sections of A are null-sets.
(b) Show that the vertical sections of A have full measure.
(c) Show that A is non-measurable with respect to 2-dimensional Lebesgue measure.

11.6 Differentiation

11.6.1. (2) What is the Radon–Nikodym derivative of the Lebesgue measure?

11.6.2. (3) Assume that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, and $\forall x, y \ |f(x) - f(y)| \leq K|x - y|$.
(a) Show that f is the integral-function of a Lebesgue-measurable g.
(b) Show that $|g| \leq K$ a.e.

11.6.3. (5) True or false? If f is absolutely continuous and strictly increasing on $[a, b]$, then its inverse is also absolutely continuous.

Answer→

11.6.4. (4) Prove that if f and g are absolutely continuous on $[a, b]$, then $f \cdot g$ is also absolutely continuous on $[a, b]$.

11.6.5. (5) Let $f : C \to [0, 1]$ be the Cantor function. For each $H \subset [0, 1]$ Borel set let $\mu_1(H) = \lambda(f(H \cap C))$, $\mu_2(H) = \lambda(f^{-1}(H))$ and $\mu_3 = \mu_1 + \mu_2$. Which pairs of the measures μ_1, μ_2, μ_3 and λ are singular, absolutely continuous? What is the Lebesgue decomposition of the measures μ_i with respect to Lebesgue measure? What is the Lebesgue decomposition of Lebesgue measure with respect to the μ_i?

11.6.6. (7) Construct a strictly increasing singular function on $[0, 1]$.

11.6.7. (9) $f : [0, 1] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0, 1]$. Show that for all $\varepsilon > 0$ the graph of f can be covered with countably many rectangles (not necessarily parallel to the axis) in such a way that the sum of the shorter sides is less than ε.

(Vojtech Jarnik competition, 2010)

Chapter 12

Complex differentiability

12.0.1 Complex numbers

12.0.1. (3)

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots = ?$$

Hint→

12.0.2. (3) Let $a, b, c \in \mathbb{C}$. What is the geometric interpretation of

$$\frac{1}{2} \operatorname{Im}\Big((c-a) \cdot \overline{(b-a)}\Big)?$$

Answer→

12.0.3. (4) Assume that $w : \mathbb{C} \to \mathbb{C}$ is a distance preserving map. Show that $w(z) = Az + B$ or $w(z) = A\bar{z} + B$, where $|A| = 1$.

12.0.4. (2) What are the product, the sum and the sum of squares of the complex mth roots of unity?

Hint→

12.0.5. (5) What is the product, the sum, and the sum of squares of all primitive m-th roots of unity?

12.0.6. (3) Let $A_1 A_2 \ldots A_n$ be the vertices of a regular n-gon, inscribed into a unit circle, and let P be another point on the circle. Prove that

$$PA_1 \cdot PA_2 \cdot \ldots \cdot PA_n \le 2.$$

12.0.7. (5) Let $p(z) \in \mathbb{C}[z]$ be of degree at least 1. Prove the following

(a) If all roots of p have negative real parts, then $\operatorname{Re} \dfrac{p'(z)}{p(z)} > 0$.

(b) If the roots of $p(z)$ all lie in the half plane $\operatorname{Re} z < 0$, then the same holds for $p'(z)$.

(c) (Gauss) If $p(z) \in \mathbb{C}[z]$, then the roots of p' are contained in the convex hull of the roots of p.

12.0.8. (7) Let $f(z) \in \mathbb{C}$ be non-constant. Prove the following

(a) $\operatorname{Re} f$ and $\operatorname{Im} f$ have no local extrema.

(b) If $|f|$ has a local extremum at z_0, then $f(z_0) = 0$.

(c) Prove the fundamental theorem of algebra.

12.0.9. (7) Let $n \geq 2$ and $u_1 = 1, u_2, \ldots, u_n$ be complex numbers with absolute value at most 1, and let

$$f(z) = (z - u_1)(z - u_2) \ldots (z - u_n).$$

Show that the polynomial $f'(z)$ has a root with non-negative real part.

KöMaL A. 430.

Solution→

12.0.10. (3) Let $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ be the so-called Zhukowksy map. What is the image of

(a) the unit circle?

(b) the interior of the unit circle?

(c) the exterior of the unit circle?

(d) the circles with center 0?

(e) the lines passing through 0?

Answer→

Related problem: 12.1.1

12.0.11. (3) Sketch the set of those complex numbers for which

(a) $\left|\dfrac{z-1}{z+1}\right| = 1$; (b) $\left|\dfrac{z-1}{z+1}\right| = 2$;

(c) $\arg(z+1) = \arg(2z-1) \quad (-\pi < \arg z \leq \pi)$.

12.0.12. (3) Sketch the set of those complex numbers for which

(a) $\operatorname{Re}(z^2) = 4$; (b) $\operatorname{Re} \dfrac{z-1}{z+1} = 0$; (c) $0 < \operatorname{Re}(iz) < 2\pi$;

(d) $|\arg(z)| < \dfrac{\pi}{4}$.

12.0.13. (3) Sketch the set of those complex numbers for which

(a) $\dfrac{|z|}{\operatorname{Re} z} < K$; (b) $|z-1| + |z+1| < 4$; (c) $\operatorname{Re} \dfrac{1+z}{1-z} > 0$.

12.0.14. (7) Let $k(z) = \dfrac{z}{(1-z)^2}$ be the so-called Koebe map. What is the image of the unit disc under the Koebe map?

12.0.15. (8) Let $f \in \mathbb{C}[x]$ and let T be a rectangle such that f has no root on the boundary of T. Show that the number of roots of f inside T agrees with the winding number about 0 of the image of the boundary of T under f.

12.0.16. (5) Let $m > 1$ and $a, b : \mathbb{Z}_m \to \mathbb{C}$ be two functions. Define the sum $a + b$ and the convolution $a * b$ of a and b as follows

$$(a+b)(n) = a(n) + b(n); \qquad (a*b)(n) = \sum_{k=0}^{m-1} a(k)b(n-k).$$

Prove that this makes the set of complex valued functions on $\mathbb{Z}_m$ a commutative ring with unit.

12.0.17. (6) Let $\varepsilon = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$. Define the Fourier transform of a function $a : \mathbb{Z}_m \to \mathbb{C}$ by

$$\hat{a}(n) = \sum_{k=0}^{m-1} a(k)\varepsilon^{nk}.$$

Show that $\widehat{(a*b)}(n) = \hat{a}(n) \cdot \hat{b}(n)$.

12.0.18. (8) Find a formula for Fourier inversion in case of the finite Fourier transform.

12.0.19. (9) Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function for which $\lim\limits_{z\to\infty} \dfrac{f(z)}{z} = 1$ (i.e. $\dfrac{f(z)}{z} \to 1$ if $|z| \to \infty$). Show that the image of f is $\mathbb{C}$.

12.0.20. (6) Let $a_1, a_2, \ldots$ be a decreasing sequence of positive numbers that converges to 0, and let $b_1, b_2, \ldots$ be a sequence of complex numbers such that the partial sums $b_1 + \ldots + b_n$ are bounded by a constant independent of n. Prove that $\sum\limits_{n=1}^{\infty} a_n b_n$ is convergent.

12.0.21. (9) Consider $\mathbb{C}$ as the xy-plane in 3-space and pick 2 semicircles in the upper half space whose end points are the complex numbers a, b and c, d. Show that the two semicircles intersect each other orthogonally if and only if $(a, b, c, d) = -1$.

(Riesz competition, 1988)

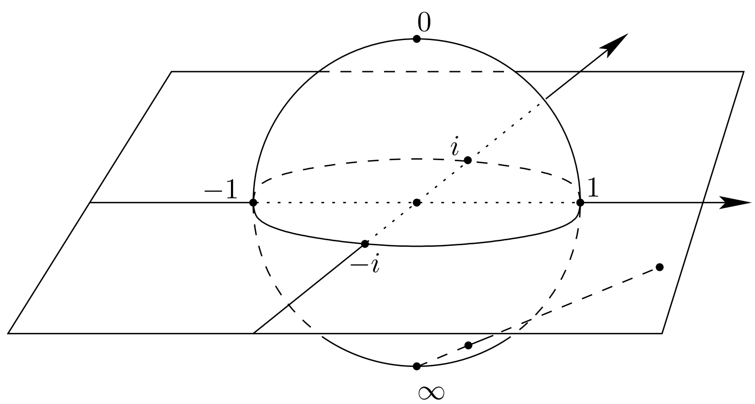
12.0.2 The Riemann sphere

12.0.22. (9) Stereographic projection (see figure) gives a bijection between points on the unit sphere and the set $\mathbb{C} \cup \{\infty\}$.

(a) Under this identification what transformations of the sphere arise from the following complex functions?

$$z \mapsto -z; \qquad z \mapsto \overline{z}; \qquad z \mapsto iz; \qquad z \mapsto \frac{1}{z}; \qquad z \mapsto \frac{-1}{\overline{z}}; \qquad z \mapsto \frac{z-i}{1-iz}$$

(b) What complex functions correspond to rotations of the sphere?



12.1 Regular functions

12.1.1 Complex differentiability

12.1.1. (6) Apply the conformal property of complex differentiable functions to the Zhukowsky map to show that the ellipses and hyperbolas with foci -1 and 1 intersect each other orthogonally.

Related problem: 12.0.10

12.1.2. (3) At what complex numbers is $\operatorname{Im} z \cdot \operatorname{Re}^2 z \cdot i + \overline{z}$ differentiable?

12.1.3. (3) At what complex numbers is $\mathrm{Im}^2 z + \mathrm{Re}\, z + \overline{z}$ differentiable?

12.1.4. (3) At what complex numbers is $|z|^2 - (2+i)\overline{z}$ differentiable?

12.1.5. (3) Do these functions satisfy the Cauchy–Riemann equations?

$$(x^2+y^2, 2xy); \qquad (x^2-y^2, 2xy); \qquad (e^x \cos y, e^x \sin y).$$

12.1.6. (3) Show that $f(x,y) = \sqrt{|xy|}$ is not differentiable at 0 even though it satisfies the Cauchy–Riemann equations there.

12.1.7. (5) Let f be regular on the domain D with image D'. Assume that f is injective and let the area of D' be $A(D')$.

(a) Prove that

$$A(D') = \int_D |f'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

(b) Compare with the theorem on $\mathbb{R}^2 \to \mathbb{R}^2$ functions.

12.1.2 The Cauchy–Riemann equations

12.1.8. (4) Show that if $f(z)$ is differentiable at z_0, then so is $g(z) := \overline{f(\overline{z})}$ at $\overline{z_0}$.

12.1.9. (4) If f is entire, then so is $g(z) := \overline{f(\overline{z})}$.

12.1.10. (5) Let $D \subset \mathbb{R}^2$ be an open domain and $u, v : D \to \mathbb{R}^2$ twice differentiable for which the map $x + yi \mapsto u(x,y) + iv(x,y)$ is regular on D. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

12.2 Power series

12.2.1 Domain of convergence

12.2.1. (3) What is the radius of convergence of the series $\sum_0^\infty \frac{(n^2-n)!}{3^{n^2}} z^n$?

12.2.2. (4) Show that if f is the sum of a power series that converges on a disc of radius R around z_0, then the average of f around a circle of radius $r < R$ centered at z_0 is $f(z_0)$.

12.2.3. (4) For which $z \in \mathbb{C}$ is $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}(z+2i)^n$ convergent?

12.2.4. (4) For which $z \in \mathbb{C}$ is $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}(z+1-2i)^n$ convergent? Absolutely convergent?

12.2.5. (4) Find the Taylor series of $1/(z^2-1)$ around $-2i$ and determine its radius of convergence.

12.2.6. (4) Find the Taylor series of $1/z$ around i and determine its radius of convergence.

12.2.7. (4) Find the Taylor series of $1/(z^2-1)$ around i and determine its radius of convergence.

12.2.8. (3) Find the radius of convergence of the following series. At which points do they converge, do they converge absolutely? What is their termwise derivative, antiderivative and what is the radius of convergence of those series? What is the largest disc with the same center as the power series to which these functions extend as regular functions?

$$\sum_{n=0}^{\infty} z^n; \qquad \sum_{n=0}^{\infty}(n+1)(z+1)^n \qquad \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!}; \qquad \sum_{n=1}^{\infty} \frac{(z+i)^n}{n}.$$

12.2.9. (5) (a) $f(z) = \displaystyle\sum_{0}^{\infty} \frac{z^n}{n}$ converges at all points on the unit circle except $z = 1$.

(b) The function can be analytically continued along any of these points.

12.2.2 Regularity of power series

12.2.10. (6) Assume that $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is convergent in the unit disc and is injective there. Express the area of the image of the unit disc in terms of the coefficients a_n.

12.2.11. (6) [(Parseval formula for power series)] Assume that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is convergent on the disc $|z| < r + \varepsilon$. Prove that

$$\frac{1}{2\pi r} \int\limits_{|z|=r} |f(z)|^2 \cdot |\,\mathrm{d}z| = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

12.2.3 Taylor series

12.2.12. (5) Find the first four terms of the Taylor series around 0 of the following functions:

a) $\tan z$ b) $\dfrac{1}{e^z - 1}$ c) e^{e^z} d) $\dfrac{e^z - 1}{\sin z}$

12.3 Elementary functions

12.3.1 The complex exponential and trigonometric functions

12.3.1. (7) Let $f(0) = 0$ and $f(z) = \dfrac{1}{\sin z} - \dfrac{1}{z}$ when $z \neq 0$. Is f differentiable at 0?

12.3.2. (4) Show that the only periods of $\sin z$ are $2k\pi$, for k an integer.

12.3.3. (6) Let D_ε be the domain that one gets by deleting discs with center $k\pi$ ($k \in \mathbb{Z}$) and radius $\varepsilon < \pi/2$. Show that both $1/\sin z$ and $\cot z$ are bounded on D_ε.

12.3.4. (3) Does e^{-1/z^4} have a limit at 0?

12.3.5. (5) Does any of the functions e^{iz}, $\sin z$, $\cos z$, $\tan z$, $\cot z$ have a limit as $\operatorname{Im} z \to \pm\infty$?

12.3.6. (3) Prove that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

and

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

12.3.7. (4) Use the Cauchy product of the series that define the complex exponential to show that $e^{z+w} = e^z e^w$.

12.3.8. (3) Prove that the following equations have only real roots a) $z \sin z = 1$ b) $\tan z = z$.

12.3.2 Complex logarithm

12.3.9. (5) If f is regular and non-vanishing on the star-shaped domain D prove that the antiderivative of f'/f defines $\log f$ as a regular function on D.

12.3.10. (5) Let $c \in \mathbb{C}$ and for $\operatorname{Re} z > -1$ let $f(z) = (1+z)^c = \exp\big(c \cdot \log(1+z)\big)$, where log is the principal branch. For what c can f be continued through -1?

12.3.11. (4) Take the branch of logarithm on $\mathbb{C} \backslash \{x + iy : x \geq 0, y = \sin x\}$ for which $\log 1 = 0$. What is $\log(e^{3/2})$ for this branch?

12.3.12. (4) What are the possible values of

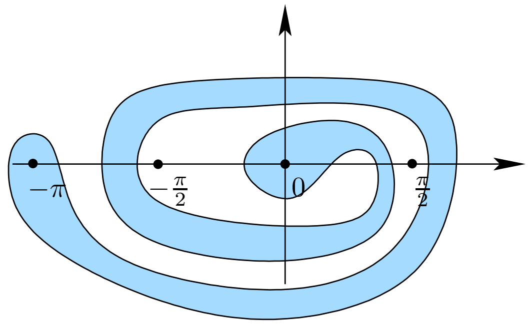
$$e^{\pi e^{i\pi/2}} \qquad \log(3 + \sqrt{3}i)?$$

12.3.13. (6) (a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is continuous and non-vanishing, then $\arg f$, $\log f$, f^α (for any $\alpha \in \mathbb{C}$) can be defined as continuous functions on $\mathbb{C}$.

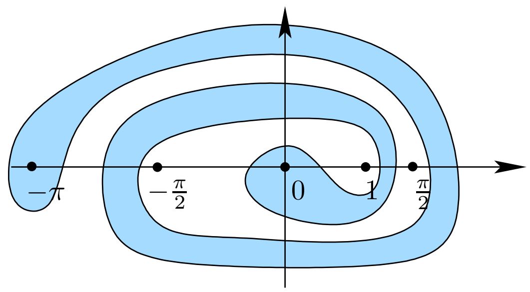
(b) Prove the fundamental theorem of algebra using the function $z + c\sqrt[n]{p(z)}$ and the Brouwer fixed-point theorem.

12.3.14. (8) Can one prove the fundamental theorem of algebra by applying the Brouwer fixed-point theorem to $z + af(bz + c)$ with suitable a, b, c?

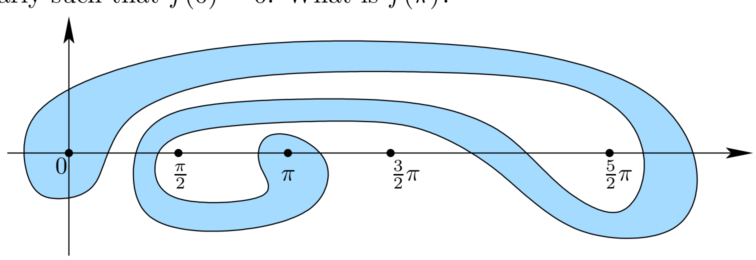
12.3.15. (9) On the domain in the figure $f(z) = \sqrt[3]{\cos z}$ can be defined regularly such that $f(0) = 1$. What is $f(-\pi)$?



12.3.16. (9) On the domain in the figure $f(z) = \sqrt{\dfrac{\cos z}{1-z}}$ can be defined regularly such that $f(0) = 1$.What is $f(-\pi)$?



12.3.17. (9) On the domain in the figure $f(z) = \log \cos z$ can be defined regularly such that $f(0) = 0$. What is $f(\pi)$?



12.3.18. (6) Sketch the following sets of the complex plane:

$$\left\{e^z:\ 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < \frac{\pi}{2}\right\}; \qquad \left\{\log \frac{1-z}{1+z}:\ \ \operatorname{Re} z > 0\right\};$$

$$\left\{\cos z:\ 0 < \operatorname{Re} z < \frac{\pi}{2},\ 0 < \operatorname{Im} z\right\}; \qquad \left\{\sin z:\ 0 < \operatorname{Re} z < \frac{\pi}{2},\ 0 > \operatorname{Im} z\right\}.$$

12.3.19. (5) Determine the image of the following maps:

a) $w(z) = \log z \qquad D = \mathbb{C} \backslash (-\infty, 0]$
b) $w(z) = \log z \qquad D = \{|z| > 1,\ \operatorname{Im} z > 0\}$
c) $w(z) = \tan z \qquad D = \{0 < \operatorname{Re} z < \pi\}$
d) $w(z) = \cot z \qquad D = \{0 < \operatorname{Re} z < \pi/4\}$
e) $w(z) = \sin z \qquad D = \{0 < \operatorname{Re} z < 2\pi,\ \operatorname{Im} z > 0\}$

12.3.20. (4) At which points is the regular branch of $\log(1 + z)$ differentiable? What are the Taylor coefficients at 0? At 1? What is the radius of convergence?

Chapter 13

The Complex Line Integral and its Applications

13.0.3 The complex line integral

13.0.1. (4) Find the following integrals:

a) $\displaystyle\int_{|z|=1} \operatorname{Im}(z)\,\mathrm{d}z$ b) $\displaystyle\int_{|z|=1} \overline{z}\,\mathrm{d}z$ c) $\displaystyle\int_{[0,1+i]} e^z\,\mathrm{d}z$

d) $\displaystyle\int_{|z|=1} \frac{1}{z}\,\mathrm{d}z$ e) $\displaystyle\int_{[1,i]} |z|^2\,\mathrm{d}z$ f) $\displaystyle\int_{|z|=2} \frac{1}{z^2+1}\,\mathrm{d}z$

13.0.2. (3) Let Γ_1 be the union of $(0,1)$ and $(1,1+i)$ oriented from 0 to $1+i$, let Γ_2 be the segment from 0 to $1+i$ and let Γ_3 be the parabolic arc on $\operatorname{Im} z = (\operatorname{Re} z)^2$ from 0 to $1+i$. Calulate $\int_{\Gamma_j} z^2$ from the definition.

13.0.3. (3) Find the following integrals:

$$\int_{|z|=1} \operatorname{Im} z \cdot \operatorname{Re} z\,\mathrm{d}z; \qquad \int_{|z|=1} \overline{z}\,\mathrm{d}z; \qquad \int_{[1,i]} |z|^2\,\mathrm{d}z.$$

13.0.4. (3) Let γ be the parabolic arc on $\operatorname{Im} z = (\operatorname{Re} z)^2$ from $-1+i$ to $1+i$.

$$\int_\gamma |z|^2\,\overline{\mathrm{d}z} = ?$$

13.0.5. (3) Let Γ be the parabolic arc on $\operatorname{Im} z = (\operatorname{Re} z)^2$ from 0 to $1+i$. Find the following integrals:

$$\int_\Gamma z^2 \, \mathrm{d}z; \qquad \int_\Gamma z^2 |\, \mathrm{d}z|; \qquad \int_\Gamma z^2 \, \overline{\mathrm{d}z}; \qquad \int_\Gamma |z^2| \cdot |\, \mathrm{d}z|; \qquad \int_\Gamma |z^2| \cdot \operatorname{Im} \, \mathrm{d}z.$$

For which ones can the fundamental theorem of calculus of complex line integrals be applied?

13.0.6. (3) Determine the complex line integral of $1/z$ along a positively oriented circle of center 0 with radius r.

13.0.7. (3) Let $r > 0$ and $n \in \mathbb{Z}$. Find $\displaystyle\int_{|z|=r} z^n \, \mathrm{d}z$.

13.0.8. (7) Of the roots of the polynomial $p(z)$, k is in $\{z : |z| < r\}$; the others are outside. Let $\gamma(t) = p(re^{it})$ $(0 \le t \le 2\pi)$.

(a) How can $\int_\gamma \frac{\mathrm{d}z}{z}$ be computed using a substitution?

(b) What is the index of γ around 0?

13.0.9. (7) Let $D \subset \mathbb{C}$ simply connected and $f : D \to \mathbb{C}$ univalent. Prove that $f(D)$ is also simply connected.

13.0.4 Cauchy's theorem

13.0.10. (7) Show that for all $a \in \mathbb{C}$

$$\int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{iax} \, \mathrm{d}x = \sqrt{2\pi} \cdot e^{-a^2/2}.$$

13.0.11. (5) Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ have degree $n > 1$ and no roots in $|z| > R$. Let $I(R) = \dfrac{1}{2\pi i} \displaystyle\int_{|z|=R} \frac{dz}{p(z)}$. Show that

(a) $\displaystyle\lim_{R\to\infty} I(R) = 0$; (b) $I(R)$ is constant. (c) $I(R) = 0$.

13.0.12. (5) Find the following integrals:

a) $\displaystyle\int_{[0,1+i]} e^z dz$ b) $\displaystyle\int_{|z|=1} \frac{1}{z} dz$ c) $\displaystyle\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1}$

(T is the square with vertices $\pm 1 \pm i$ oriented positively.)

13.0.13. (6) Let D be a simply connected domain that does not contain the origin.
(a) Show that $1/z$ has an antiderivative on D.
(b) Show that if $g'(z) = 1/z$ on D, then $ze^{-g(z)}$ is constant.
(c) Show that $\log z$ has a continuous branch on D.

13.0.14. (6) Let D be a simply connected domain and $f(z)$ a non-vanishing holomorphic function on D.
(a) Show that $f'(z)/f(z)$ has an antiderivative on D.
(b) Show that if $g' = f'/f$ on D, then $f(z)e^{-g(z)}$ is constant on D.
(c) Show that $\log f$ has a continuous branch on D.

13.0.15. (5) Let a and b be different complex numbers. Show that on $\mathbb{C} \setminus [a, b]$ there is a holomorphic branch of $\log \frac{z-a}{z-b}$.

13.1 The Cauchy formula

13.1.1. (8) Let f be a holomorphic function on the disc $|z| < 1 + \varepsilon$ and let $|a| < 1$. Find a function $\varphi_a : [0, 2\pi] \to \mathbb{R}$ such that

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \varphi_a(t) dt.$$

13.1.2. (8) Prove for any complex number a that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| e^{it} + a \right| \mathrm{d}t = \begin{cases} \log |a| & \text{if } |a| > 1, \\ 0 & \text{if } |a| \leq 1. \end{cases}$$

13.1.3. (6) Let f be continuous on the closed unit disc and holomorphic in its interior. Prove that for $|z| < 1$

$$f(z) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(\xi)}{z - \xi} \mathrm{d}\xi.$$

13.1.4. (8) Let f be a holomorphic function on the disc $|z| < 1 + \varepsilon$. Prove that

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \mathrm{d}t.$$

When does equality hold?

13.1.5. (7) Let $n \in \mathbb{Z}$. Find

$$\int_{|z|=2} \frac{z^n}{(z-1)(z-3)}\,\mathrm{d}z.$$

13.1.6. (4)

$$\frac{1}{2\pi i}\int_{|z|=5} \frac{\cos z}{z}\,\mathrm{d}z = ? \qquad \int_{|z|=3} \frac{e^z}{z}\,\mathrm{d}z = ? \qquad \int_{|z|=3} \frac{e^z}{z-2}\,\mathrm{d}z = ?$$

$$\int_{|z|=3} \frac{e^z}{(z-2)(z-4)}\,\mathrm{d}z = ?$$

13.1.7. (7) Let $a, b \in \mathbb{C}$ and $|b| < 1$. Prove that

$$\frac{1}{2\pi}\int_{|z|=1} \left|\frac{z-a}{z-b}\right|^2 |\,\mathrm{d}z| = \frac{|a-b|^2}{1-|b|^2} + 1.$$

Hint→ Solution→

13.1.8. (2)

$$\int\limits_{|z|=2} \frac{3^z}{(z-1)^2(z+3)^2}\,\mathrm{d}z = ?$$

13.1.9. (2) The function $f(z)$ is holomorphic in the interior of the unit disc ($|z| < 1$) and $|f| < 1$. How large can $|f'''(0)|$ be?

Answer→

13.1.10. (5) Show that if $f \in O(|z| \leq 1)$, then a) $f'(z)\,(1-|z|)$ is bounded.

b) What can we say about the n-th derivative?

13.1.11. (3)

$$\frac{1}{2\pi i}\int_{|z|=5} \frac{\cos z}{z^2}\,\mathrm{d}z = ? \qquad \int_{|z|=3} \frac{e^z}{z^8}\,\mathrm{d}z = ? \qquad \int_{|z|=3} \frac{e^z}{(z-2)^3}\,\mathrm{d}z = ?$$

13.1.12. (3) For $a, r > 0$ find the following integrals:

$$\frac{1}{2\pi i}\int\limits_{|z|=r} a^z \,\mathrm{d}z; \quad \frac{1}{2\pi i}\int\limits_{|z|=r} \frac{a^z}{z} \,\mathrm{d}z; \quad \frac{1}{2\pi i}\int\limits_{|z|=r} \frac{a^z}{z+1} \,\mathrm{d}z; \quad \frac{1}{2\pi i}\int\limits_{|z|=r} \frac{a^z}{z^2} \,\mathrm{d}z;$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r} \frac{a^z}{(z+2)^2} \,\mathrm{d}z.$$

13.2 Power and Laurent series expansions

13.2.1 Power series expansion and Liouville's theorem

13.2.1. (9)

The sequence $a_0, a_1, \ldots,$ is defined recursively by $a_0 = -1$ and the requirement $\sum_{k=0}^{n} \frac{a_k}{n-k+1} = 0$ for all $n \geq 1$. Show that for all $n \geq 1$ $a_n > 0$. (IMO Shortlist, 2006)

Use complex analysis to solve this probem by showing that

$$a_n = \int_1^\infty \frac{\mathrm{d}x}{x^n(\pi^2 + \log^2(x-1))}.$$

13.2.2. (5) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be entire that satisfies $|f(z)| < e^{|z|}$. Prove that $|a_n| \leq \left(\frac{e}{n}\right)^n$.

13.2.3. (9) Prove that if f is entire and its image is disjoint from the real interval $[-1, 1]$, then f is constant.

Related problem: 12.0.10

13.2.4. (7) Show that if f is a double peridodic entire function (i.e. $f(z+a) = f(z), f(z+b) = f(z)$ where a and b are linearly independent over $\mathbb{Q}$, then f is constant.

13.2.5. (4) Let $f \in O(\mathbb{C})$. Then $\operatorname{Re} f$ cannot be bounded either from below or above.

13.2.6. (3) Find the Taylor series of $\dfrac{z^2+i}{z^2+z}$ around i.

13.2.7. (5) Find the Taylor series of $(1+x)^c = \exp\big(c \cdot \log(1+z)\big)$ around 0.

13.2.8. (4) Describe those $f \in O(\mathbb{C})$ which do not take positive values.

13.2.9. (6) Assume that $f : \mathbb{C} \leftrightarrow \mathbb{C}$ is a biholomorphism. Show that $f(z) = Az + B$.

13.2.2 Laurent series

13.2.10. (6) Assume that f has antiderivatives of all order on the set $1 < |z| < 2$. Show that f has an analytic continuation to $|z| < 2$.
Related problem: 14.2.4

13.2.11. (5) (Parseval formula for Laurent series) Assume that $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ converges on $r - \varepsilon < |z| < r + \varepsilon$. Prove that

$$\frac{1}{2\pi r} \int\limits_{|z|=r} |f(z)|^2 \cdot |\,\mathrm{d}z| = \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}.$$

13.2.12. (5) Find the Laurent series of $\dfrac{e^z}{z-1}$ around 0 on $|z| > 1$.

13.2.13. (7) Compute the coefficients of the Laurent expansion of $f(z) = \dfrac{1}{(z-2)(z+1)}$ on $1 < |z| < 2$ by using the Cauchy formula.

13.2.14. (3) Find the Laurent series of $\dfrac{2z^3-1}{z^2+z}$ around i, on $1 < |z-i| < \sqrt{2}$.

13.2.15. (3) Find the Laurent series of $z \mapsto \dfrac{z}{z^2-3z+2}$ around 3 on $|z-3| < 1$, $|z-3| > 2$ and $1 < |z-3| < 2$.

13.2.16. (3) Find the Laurent series of $\dfrac{1}{1-z}$ in $1 < |z-2| < 3$.

13.2.17. (3) Find the Laurent series of $\dfrac{1}{1-z}$ around 3 (within a disc of radius 2).

13.2.18. (5) Find the Laurent series of $e^{z+1/z}$ around 0.

13.3 Local properties of holomorphic functions

13.3.1 Consequences of analyticity

13.3.1. (3) An entire function $f(z)$ satisfies $|f(1/n)| = 1/n^2$ for $n = 1, 2, \ldots$, and $|f(i)| = 2$. What are the possible values of $|f(-i)|$?

Hint→ Solution→

13.3.2. (7) Show that if f takes only real values on the real and imaginary axes, then f is even.

Hint→

13.3.3. (5) Give an example of a function that is holomorphic in the open unit disc and has infinitely many roots there.

Solution→

13.3.4. (6) Assume that $f \in O(\mathbb{C})$ and $|f(x)| = 1$ for all $x \in \mathbb{R}$. Prove that $\overline{f(\bar{z})} = \dfrac{1}{f(z)}$.

13.3.5. (7) If $f \in O(|z| > 1)$, is bounded and $f(n) = 0 \qquad (n = 2, 3, \ldots)$, then $f \equiv 0$.

13.3.6. (7) Show that if $f \in O(\mathbb{C})$, $\left| f\left(\dfrac{1}{n}\right) \right| < \dfrac{1}{2^n}$, then $f \equiv 0$. Can one do better?

13.3.7. (8) Given that $f \in O(\mathbb{C})$, $f\left(\dfrac{1}{n^2}\right) = \cos\dfrac{1}{n}$ find $f(-1)$.

13.3.2 The maximum principle

13.3.8. (7) Let f be continuous on the closed unit disc and holomorphic inside. Let $A = \max\limits_{0 \le t \le \pi} |f(e^{it})|$ and $B = \max\limits_{\pi \le t \le 2\pi} |f(e^{it})|$. Show that $|f(0)| \le \sqrt{AB}$.

13.3.9. (5) Let f be continuous on the closed unit disc and holomorphic inside. Show that the image of the open disc is in the convex hull of the image of the boundary circle.

13.3.10. (5) Prove that if f is holomorphic on an open set, then neither the real part nor the imaginary part of f has a local extrema.

13.3.11. (9) [(Hadamard)] Let $0 < r_1 < r_2 < r_3$ and let f be holomorphic on $r_1 < |z| < r_3$ with a continuous extension to the boundary. Prove that

$$\left(\max_{|z|=r_2} |f(z)|\right)^{\log(r_3/r_1)} \leq \left(\max_{|z|=r_1} |f(z)|\right)^{\log(r_3/r_2)} \left(\max_{|z|=r_3} |f(z)|\right)^{\log(r_2/r_1)}.$$

13.4 Isolated singularities and residue formula

13.4.1 Singularities

13.4.1. (4) Prove that $\frac{z}{\sin z}$ and $\frac{1}{\sin z} - \frac{1}{z}$ have removable singularities at 0.

13.4.2. (5) Assume that f has a pole of order m at a and that p is a polynomial of degree n. Show that $p(f(z))$ has a pole of order mn at a.

13.4.3. (7) Can e^f have a pole at a point where f has an isolated singularity?

13.4.4. (4) Show that if f is holomorphic and bounded on $|z| > 1$, then it has a limit at ∞.

13.4.2 Cauchy's theorem on residues

13.4.5. (4) If $f \in \mathcal{M}(|z| < 1)$, then f has an antiderivative if and only if the residue of f is 0 at all singularities.

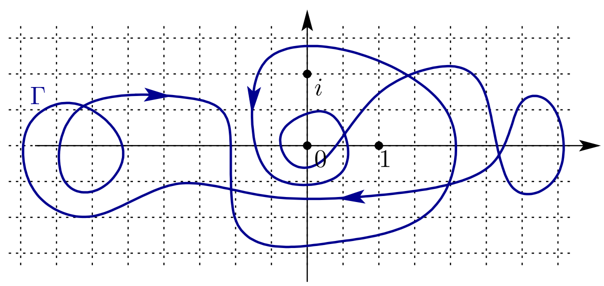
13.4.6. (5) Calculate the first 6 terms in the Laurent series of $\cot z$ and $\pi \cot(\pi z)$ on the domain $0 < |z| < \pi$. What are the residues of $\frac{\cot z}{z}$, $\frac{\cot z}{z^2}$, $\ldots$, $\frac{\cot z}{z^5}$ in 0?

13.4.7. (4)

$$\frac{1}{2\pi i}\int_{|z|=2} \tan z \, \mathrm{d}z = ?$$

13.4.8. (4)

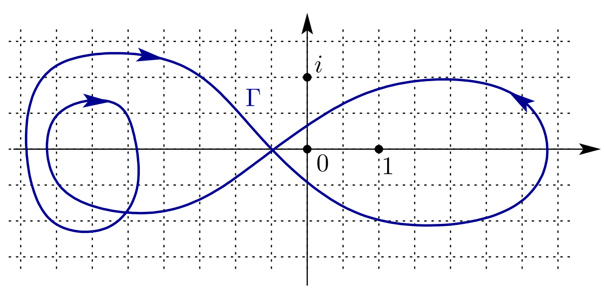
$$\int_{\Gamma} \frac{\tan z}{z^2+1} \, \mathrm{d}x = ?$$



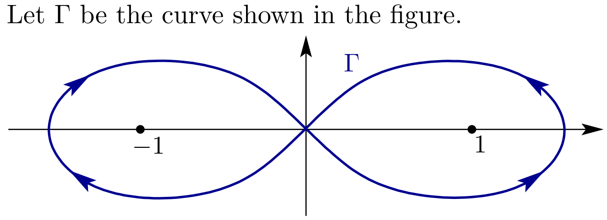
13.4.9. (3) What are the singularities of $\pi \cot \pi z$? Find the residues at these points.

13.4.10. (4)

$$\int_{\Gamma} \frac{\mathrm{d}z}{\cos z} = ?$$



13.4.11. (4) Let Γ be the curve shown in the figure.

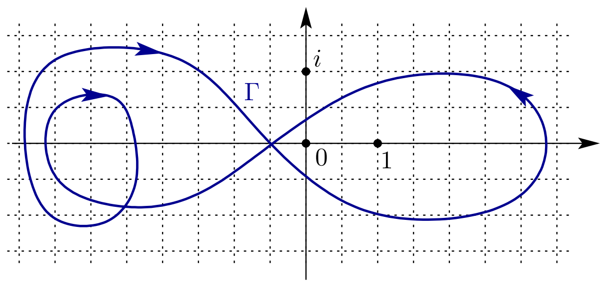


(a) Compute $\displaystyle\int_{\Gamma} \frac{z^{20}+2}{z^2-1}\,\mathrm{d}z$.

(b) Compute $\displaystyle\int_{C(0,1)} \frac{\sin z}{z}\,\mathrm{d}z$.

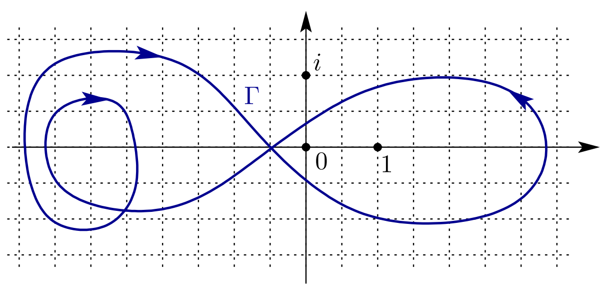
13.4.12. (4)

$$\int_{\Gamma} \frac{\mathrm{d}z}{(z-1)^2 \sin z} = ?$$



13.4.13. (5)

$$\int_{\Gamma} \frac{\mathrm{d}z}{\left(z-\frac{\pi}{3}\right)^2 \sin z} = ?$$



13.4.14. (5)

$$\frac{1}{2\pi i} \int_{|z|=1/4} \frac{\mathrm{d}z}{\sin \frac{1}{z}} = ?$$

13.4.15. (4)

$$\int_{|z|=2} \frac{\sin \frac{\pi}{z}}{z^4-1} = ?$$

13.4.16. (4) Let $0 < r < \pi$. $\displaystyle\int\limits_{|z|=r} \frac{\mathrm{d}z}{\sin z} = ?$

13.4.17. (7) Show that if the complex numbers $a_1, \ldots, a_n$ are all different and $p(z) = (z - a_1) \cdot \ldots \cdot (z - a_n)$, then

$$\sum_{j=1}^{n} \frac{p''(a_j)}{(p'(a_j))^3} = 0.$$

13.4.18. (5)

$$\int_{|z|=5} \frac{z^2}{\sin z}\, \mathrm{d}z = ?$$

13.4.19. (4)

$$\int_{|z-2|=4} \frac{z}{\sin z}\, \mathrm{d}z = ?$$

13.4.3 Residue calculus

13.4.20. (5) Find the residues of $\tan z$, $\tan^2 z$, $\tan^3 z$ in $\frac{3\pi}{2}$.

13.4.21. (5) What are the residues of $\dfrac{\tan z}{1 - \cos z}$ and $\dfrac{e^z}{\tan z - \sin z}$ in 0?

13.4.22. (4) Find the singularities and residues of the following functions:

$$\frac{1}{z}; \quad \frac{1}{z^2}; \quad \frac{1}{z^2 + 2z}; \quad \frac{1}{\sin z}; \quad \sin\frac{1}{z}; \quad \frac{e^z}{z^2+4}; \quad \frac{e^z}{(z^2+4)^2};$$

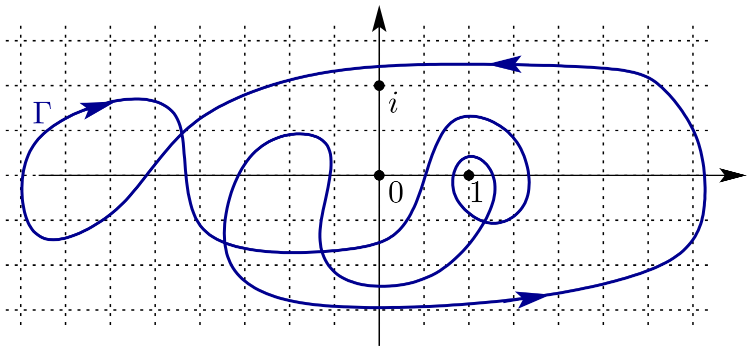
$$\frac{e^z}{(z^2+4)^3} \quad \frac{e^z - z^3 + 8}{z^2 + 1}$$

13.4.23. (5) Let f and g be holomorphic in a neighborhood of z_0.

(a) Assume that g has a simple zero in z_0. Prove that $\operatorname*{Res}_{z_0} \dfrac{f}{g} = \dfrac{f(z_0)}{g'(z_0)}$.

(b) Assume that g has a double zero in z_0. Express $\operatorname*{Res}_{z_0} \dfrac{f}{g}$ in terms of Taylor coefficients of f and g.

13.4.24. (7)



$$\int_\Gamma \frac{\cot z}{z^8 - z^6 - z^4 + z^2}\,\mathrm{d}z = ?$$

13.4.25. (4)

$$\frac{1}{2\pi i}\int_{|z|=5} \tan z\,\mathrm{d}z = ?$$

13.4.4 Applications

Evaluation of series

13.4.26. (5) Use residues to calculate $\sum_{k=1}^{\infty} \frac{1}{k^2 - \frac{1}{4}}$. Check your result using elementary methods.

13.4.27. (5)

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + k + 1} = ?$$

(The result should not contain any complex number!)

13.4.28. (5) Use residue calculus of the function $\frac{\pi\cot(\pi z)}{z^2}$ to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

13.4.29. (5)

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = ? \qquad \sum_{k=1}^{\infty} \frac{1}{k^2 - \frac{1}{4}} = ? \qquad \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} = ?$$

13.4.30. (5)

$$\sum_{k=-\infty}^{\infty} \frac{1}{2k^2 - 1} = ?$$

13.4.31. (5) Let N_k be the square with vertices $\pm(k + \frac{1}{2}) \pm (k + \frac{1}{2})i$. What is

$$\frac{1}{2\pi i} \int_{N_k} \frac{\pi \cot \pi z}{z^2}\, \mathrm{d}z?$$

What identity results if we let $k \to \infty$?

Evaluation of integrals

13.4.32. (4)

$$\int_0^{\infty} \frac{\mathrm{d}x}{x^7 + 1} = ?$$

(Simplify as much as possible.)

13.4.33. (4) Let $a \in (0, 1)$.

$$\int_0^{\infty} \frac{x^{\alpha}}{x^2 + 1}\, \mathrm{d}x = ?$$

13.4.34. (6)

$$\int_0^{\infty} \frac{\cos x}{x^2 + 1}\, \mathrm{d}x = ?$$

13.4.35. (7)

$$\int_0^{\infty} \frac{\mathrm{d}x}{x^3 + 1} = ? \qquad \int_0^{\infty} \frac{\log x}{x^2 + x + 1}\, \mathrm{d}x = ? \qquad \int_0^{\infty} \frac{\log^2 x}{x^2 + 1}\, \mathrm{d}x = ?$$

13.4.36. (6)

$$\int_0^\infty \frac{\log x}{x^3+1}\,\mathrm{d}x = ?$$

13.4.37. (6)

$$\int_0^\infty \frac{\log x}{x^2-1}\,\mathrm{d}x = ?$$

13.4.38. (5)

$$\int_{|z|=2} \frac{\mathrm{d}z}{(z^4+z^2)\sin z} = ?$$

13.4.39. (5)

$$\int_{|z|=2} \frac{\mathrm{d}z}{(z^2+1)\sin z} = ?$$

13.4.40. (9) a) $\displaystyle\int_0^\infty \cos x^2 dx = ?$ b) $\displaystyle\int_{-\infty}^\infty \sin(3x^2+1) dx = ?$

13.4.41. (7) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{1+e^t} dt = ?$ $(0 < \alpha < 1)$

13.4.42. (7) $\displaystyle\int_{-i\infty}^{i\infty} \frac{\cosh Az}{(z+1)(z+2)} dz = ?$ $(A > 0)$

13.4.43. (5) $\displaystyle\int_{-\infty}^{\infty} \frac{x^4-1}{x^6-1}\,\mathrm{d}x = ?$

13.4.44. (9) $\displaystyle\int_0^{\pi/2} \log\sin x dx = ?$

13.4.45. (7)

$$\int_{-\infty}^{\infty} \frac{(x-3)\cos x}{x^2-6x+109}\,\mathrm{d}x = ?$$

13.4.46. (6) a) $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+a^2}\,\mathrm{d}x \qquad (a>0) \qquad$ b) $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x$

13.4.47. (6)

$$\int_0^{\infty} \frac{\sqrt{x}}{x^3+1}\,\mathrm{d}x = ? \quad \int_{-\infty}^{\infty} \frac{e^{-it}}{x^4+1} = ? \quad \int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = ?$$

13.4.48. (7) Determine for any $a>0$ the value of the integral $\displaystyle\frac{1}{2\pi i}\int_{|z|=2} \frac{a^{\xi}}{1-\xi^2}\,\mathrm{d}\xi$.

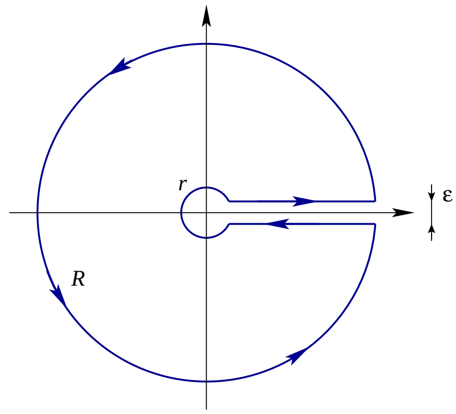
13.4.49. (7) $\displaystyle\int_{\sigma-i}^{\sigma+i} \frac{zt^z}{z^2+1}dz = ? \qquad (\sigma>0, \qquad 0<t<1)$

13.4.50. (5) a) $\displaystyle\int_{C(\pi,1)} \frac{z}{\sin z}dz = ? \qquad$ b) $\displaystyle\int_{C(\pi i,1)} \frac{e^z}{(z-\pi i)^2}dz = ?$

13.4.51. (5)
a) $\displaystyle\int_{|z-i|=1} \frac{e^{iz}}{1+z^2}dz = ? \quad$ b) $\displaystyle\int_{|z-\pi|=1} \frac{e^z}{\sin^2 z}dz = ? \quad$ c) $\displaystyle\int_{|z-2\pi i|=1} \frac{1}{e^z-1}dz = ?$
d) $\displaystyle\int_{|z|=\pi} \frac{e^z}{\cos z-1}dz = ?$

13.4.52. (5) What residues are possible for f'/f at z_0 if f has an isolated singularity in that point?

13.4.53. (6) Let $\Gamma_{r,R,\varepsilon}$ be the curve in the figure, where R is large, r is small and ε is much smaller than r. What results from the following limit? $\displaystyle\lim_{R\to\infty}\lim_{r\to+0}\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_{\Gamma_{r,R,\varepsilon}} \frac{\log z}{z^2+1}\,\mathrm{d}z$



13.4.54. (5)

$$\frac{1}{2\pi}\int_0^{2\pi}(e^{it}+e^{-it})^n dt = ?$$

13.4.55. (7) Let $a>0$. Determine

$$\int\limits_{\operatorname{Re} z=0} \frac{a^z}{z^2-1}\,\mathrm{d}z.$$

13.4.56. (4) a) $\displaystyle\int\limits_{|z|=2} \frac{z^{10}}{(z-1)^7}dz = ?$ b) $\displaystyle\int\limits_{|z|=21} \frac{1}{z(z-1)\dots(z-20)}dz = ?$

13.4.57. (9) Assume that the Dirichlet series $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ absolutely converges for $\operatorname{Re} s\ge 1$ and let $X>0$ be real. Find the following integrals:

$$\lim_{h\to\infty}\frac{1}{2\pi i}\int_{\operatorname{Re} s=1, |\operatorname{Im} s|\le h} f(z)\frac{X^s}{s} \qquad \frac{1}{2\pi i}\int_{\operatorname{Re} s=1} f(z)\frac{X^s}{s^2}$$

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=1} f(z)\frac{X^s}{s(s+1)}$$

13.4.5 The argument principle and Rouché's theorem

13.4.58. (3) How many zeros does the function $\cos z = 2z^3$ have in the unit disc?

13.4.59. (3) How many zeros do the functions have on the given domain?
(a) $\sin z = 2z^2, \qquad |z| < 1$
(b) $z^4 + z^3 - 4z + 1 = 0, \qquad 1 < |z| < 2$
(c) $z^6 - 6z + 10, \qquad |z| > 1$.

13.4.60. (3) Let $|a| = 3$. Find the number of zeros of $z^4 + z^3 + az - 1$ in the domain $1 < |z| < 2$.

13.4.61. (3) How many zeros does $2^z + 3z^2 - z$ have in the unit disc?

13.4.62. (5) Prove the fundamental theorem of algebra from Rouché's theorem.

13.4.63. (4) Prove that $az^n + 3z + 1$ has a root in the unit disc for any $a \in \mathbb{C}$.

13.4.64. (5) Let $a \in \mathbb{C}$, $|a| < 1$, $n \in \mathbb{N}$. Show that $(z-1)^n e^z = a$ has exactly n solutions in the half-plane $\operatorname{Re} z > 0$.

Chapter 14

Conformal maps

14.1 Fractional linear transformations

14.1.1. (4) (a) Prove that $(z_1, z_2, z_3) := \dfrac{z_1 - z_3}{z_2 - z_3}$ is real if and only if z_1, z_2 and z_3 are on a line.

(b) Prove that the cross-ratio $(z_1, z_2, z_3, z_4) := \dfrac{z_1 - z_3}{z_2 - z_3} : \dfrac{z_1 - z_4}{z_2 - z_4}$ is real if and only if z_1, z_2, z_3 and z_4 are on a circline.

14.1.2. (5) Prove that a map that preserves the cross-ratio is necessarily a fractional linear transfromation.

14.1.3. (3) Show that the map $1/z$ preserves cross-ratio, i.e. $(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}) = (z_1, z_2, z_3, z_4)$. Find other maps with this property.

14.1.4. (5) Show that if a map takes even one circle to a circle, then it is a fractional linear transformation.

14.1.5. (6) Assume that $f_n \in O(D)$ and $f_n \to f$ ($\neq$ *const.*) uniformly on D. Show that if for all n there is a circline K_n whose image under f_n is a circline, then f takes all circlines to circlines.

14.1.6. (7) What are the finite subgroups of the group of fractional linear transformations?

14.1.7. (7) What fractional linear transformations map the right half-plane to itself?

14.1.8. (3) What is the geometric meaning of the imaginary part of the cross ratio of four points?

14.1.9. (3) Prove using the behavior of the function at the points 0, ∞ and 1 that $\operatorname{Re} \frac{z+1}{z-1} < 0$, if $|z| < 1$.

14.1.10. (3) Prove using the behavior of the exponent at the points 0, ∞ and 1 that

$$\left| e^{\frac{z+1}{z-1}} \right| < 1 \quad (|z| < 1).$$

14.1.11. (5) (a) Prove that for all $f \in \mathbb{C}[z]$ one can find $g \in \mathbb{C}[z]$ with the property that g has no roots inside the unit disc and $|g(z)| = |f(z)|$ for $|z| = 1$.

(b) Prove the same for meromorphic functions on $\mathbb{C}$. For all meromorphic f one can find a meromorphic g which has no poles or zeros inside the unit disc and which satisfies $|g| = |f|$ on the unit circle.

14.1.12. (5) What are the possible poles and zeros of a fractional linear transformation that maps the unit circle to itself?

14.1.13. (5) What are the meromorphic functions f that satisfy $|f(z)| = 1$ for $|z| = 1$?

14.1.14. (7) Let f be regular on the disc $|z| < 1 + \varepsilon$ except for finitely many poles. Assume that $f(0) = 1$ and that the zeros and poles of f inside the unit disc listed with multiplicity are $\varrho_1, \varrho_2, \ldots, \varrho_n$, and $p_1, p_2, \ldots, p_m$ respectively. Prove that

$$\frac{1}{2\pi} \int_{|z|=1} \log |f(z)| \cdot |dz| = \log \left| \frac{p_1 p_2 \ldots p_m}{\varrho_1 \varrho_2 \ldots \varrho_n} \right|.$$

(If there are no zeros or poles, then the respective product, that is empty, is 1.)

14.1.15. (6) If the zeros of the regular $f : S(0,1) \to S(0,1)$ function are $|a_k| < 1$ complex numbers (possibly infinitely many), then

$$|f(0)| \leq \left| \prod_{i=0}^{\infty} a_i \right|.$$

14.1.16. (5) Prove the following statements.

(a) If $T(z)$ is a fractional linear transformation, then T has a fixed point in $\mathbb{C} \cup \infty$.

(b) Given $z_j,\ w_j\ \ (j = 1, 2, 3)$ with $(z_k \neq z_j,\ w_k \neq w_j)$, then there is a unique T fractional linear transformation such that $T(z_j) = w_j$.

(c) Describe the fractional linear transformations with 1, 2 or more fixed points.

14.1.17. (4) (a) Prove that all fractional linear transformations can be expressed as a composition of translations, rotations, dilations and conjugate inversion (inversion with respect to the unit circle followed by conjugation).

(b) Derive from this the basic properties of fractional linear transformations, they are bijective conformal maps of the Riemann sphere to itself that preserve the cross-ratio and circlines.

14.1.18. (5) Function f is regular on the disc $|z| < 1 + \varepsilon$. Show that

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, \mathrm{d}t.$$

14.1.19. (5) Show that there is exactly one conformal map which

(a) takes a given circle C to another circle C' in such a way that it takes 3 prescribed points on C to 3 prescribed points on C';

(b) takes a given circle C to another circle C' in such a way that it takes a prescribed point on C to a prescribed point on C' and a prescribed point inside C to a prescribed point inside C'.

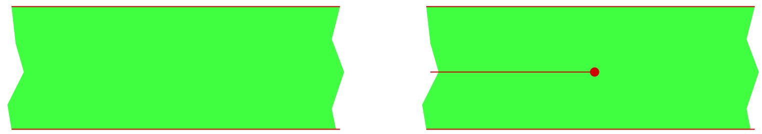
14.1.20. (5) Let H be the upper half-plane. Prove that

$$\mathrm{Aut}(H) = \left\{ T(z) = \frac{az+b}{cz+d},\ a,\ b,\ c,\ d \in \mathbb{R} \right\}!$$

If an element of $\mathrm{Aut}(H)$ is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, what matrices correspond to the same map?

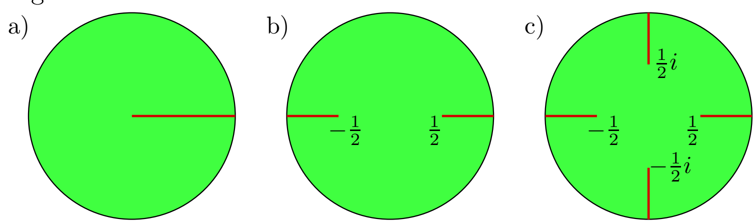
14.2 Riemann mapping theorem

14.2.1. (5) Give a biholomorphic map from $D_1 = \{z : \ |\operatorname{Im} z| < 1\}$ to $D_2 = D_1 \setminus (-\infty, 0]$.

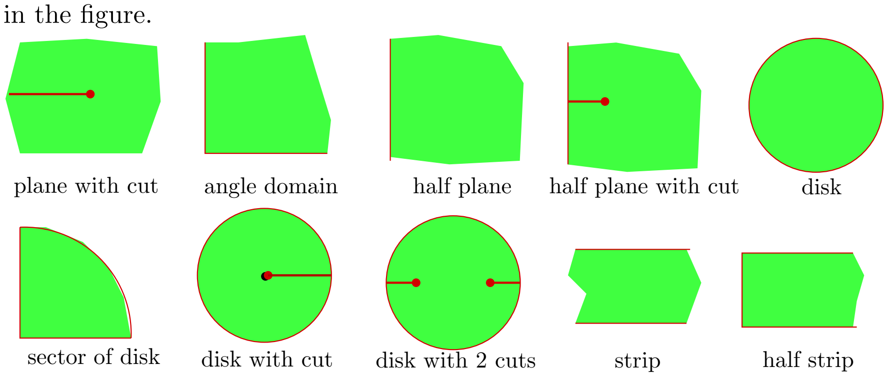


D_1 D_2

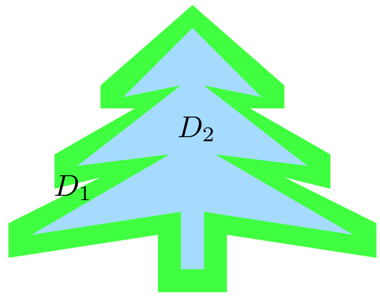
14.2.2. (7) Find a conformal bijection between the unit disc and the domain in the figure.



14.2.3. (7) Find conformal bijections between the unit disc and the domains in the figure.



14.2.4. (9) Let D_1 be the green domain in the figure, and D_2 the union of the green and blue parts. Show that if f is regular on D_1 and for all functions g that are regular on D_2 $f \cdot g$ has an antiderivative on D_1, then f can be analytically continued to D_2.



Related problem: 13.2.10

14.2.5. (7) Give a biholomorphic map from $D_1 = \{z : |\operatorname{Im} z| < 1\}$ to $D_2 = \{z : |z| < 1 \text{ and } |z - 1 - i| > 1\}$.

14.2.6. (6) Describe explicitly the comformal map in the Riemann mapping theorem for the following domains:

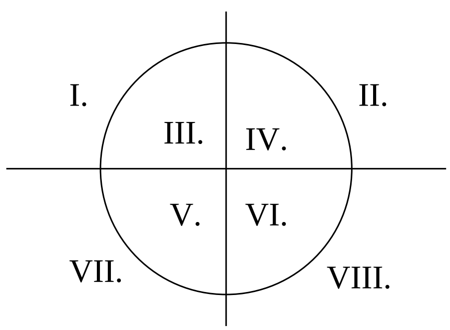
a) $\{z : -\frac{\pi}{2} < \arg z < \frac{\pi}{2}\}$ b) $\{z : |z| < 1, \ \operatorname{Im} z > 0\}$

c) $\{z : |z| < 1, \text{ or } \operatorname{Im} z < 0\}$ d) $\mathbb{C} \backslash [0, 1]$

14.2.7. (5) Let $\operatorname{Aut}(D)$ be the group of biholomorphic functions of D to itself. Show that if $f : D \leftrightarrow D'$ is a conformal bijection, then $\operatorname{Aut}(D) \cong \operatorname{Aut}(D')$.

14.2.8. (5) Let $D_1 = \{z : 0 < \operatorname{Re} z < 1, \ 0 < \operatorname{Im} z\}$ and $D_2 = \{z : \operatorname{Re} z > 0, \ \operatorname{Im} z > 0\}$. Give a formula for a biholomorphic map $D_1 \to D_2$.

14.2.9. (7) Number the domains cut by the coordinate axes and the unit circles by Roman numerals, as in the figure. Describe all biholomorphisms that permute these domains.



What possible permutations arise?

14.2.10. (5) Find conformal bijections from the domains in the figore and the upper half-plane $\operatorname{Im} w > 0$!
(a) $\{z: \ |z| > 1\} \setminus [-2, -1]$
(b) $\mathbb{C} \setminus [-1, 0] \setminus [1, \infty)$
(c) $\{z: \ |z| < 1, \ \operatorname{Im} z > 0\} \setminus \left[0, \frac{i}{2}\right]$
(d) $\{z: \ 0 < \arg z < \pi/2, \ |z| > 1\} \setminus [1+i, \infty)$

14.2.11. (7) Let $F \subsetneq G$ complex domains $f: S(0,1) \leftrightarrow F$, $g: S(0,1) \leftrightarrow G$ conformal bijections such that $f(0) = g(0)$. Show that $|f'(0)| < |g'(0)|$.

14.3 Schwarz lemma

14.3.1. (5) Let C be a circle, and p a point outside of C. Show that if f is a fractional linear transformation such that $f(C) = C$ and $f(p) = p$, then $|f'(p)| = 1$.

14.3.2. (6) For all $D \subset \mathbb{C}$ domain and $a \in D$ there is a unique $r(a, D)$ radius such that there is a conformal injection $f: D \leftrightarrow S\big(0, r(a, D)\big), \ f(a) = 0, \ f'(a) = 1$.

14.3.3. (6) Let $F \subsetneq G$ and D be complex simply connected domains $a \in F$, and $f: F \leftrightarrow D$, $g: G \leftrightarrow D$ conformal bijections such that $f(a) = g(a)$. Show that $|f'(a)| > |g'(a)|$.

14.3.4. (5) Let $P = \{z : \operatorname{Re} z > 0\}$ be the right half-plane $f: P \to P$ regular and $f(1) = 1$. Prove that $|f'(1)| \le 1$.

14.3.5. (7) Let T, $R \in \operatorname{Aut}\big(S(0,1)\big)$ and $T(a) = R(a) = 0$. Prove that $T = cR$ for some $|c| = 1$. Describe $\operatorname{Aut}\big(S(0,1)\big)$ using this observation.

14.3.6. (7) Assume that f is regular on the unit disc and satisfies $|f(z)| < 1$. Show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

14.3.7. (6) Let the roots of the regular function $f: S(0,1) \to S(0,1)$ be $a_1, \ldots, a_n$. Show that

$$|f(z)| \le \prod_{i=1}^{n} \left| \frac{a_i - z}{1 - \overline{a_i} z} \right| \qquad (|z| < 1).$$

14.3.8. (7) Assume that $f \in O(|z| < 1)$ has image $\operatorname{Re} z > 0$, and $f(0) = 1$. Show that

$$\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}.$$

14.3.9. (7) Let $w : S(0,1) \to S(0,1)$ be regular and let $|a| < 1$. Show that

a) $\left|\dfrac{w(z)-w(a)}{1-\overline{w(a)}w(z)}\right| \le \left|\dfrac{z-a}{1-\bar{a}z}\right|$ b) $|w'(a)| \le \dfrac{1-|w(a)|^2}{1-|a|^2}$.

14.3.10. (7) Let $w : S(0,1) \to S(0,1)$ be regular and $w(\alpha) = 0$. Show that

(a) $|w(z)| \le \left|\dfrac{z-\alpha}{1-\bar{\alpha}z}\right|$; (b) $|w'(a)| \le 1 - |\alpha|^2$.

14.3.11. (6) Let $a_1, a_2, \dots$ be a sequence of complex numbers such that $|a_k| < 1$ and $\operatorname{Re} a_k > \frac{1}{2}$ for all k. Let

$$z_0 = 0, \qquad z_{n+1} = \frac{z_n + a_n}{1 + \overline{a_n} z_n}.$$

Prove that $a_n \to 1$.

(based on IMC 2011/6)

14.3.12. (9) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disc and let $0 < a < 1$ be a real number. Suppose that $f : D \to \mathbb{C}$ is a holomorphic function such that $f(a) = 1$ and $f(-a) = -1$.

(a) Prove that

$$\sup_{z\in D} |f(z)| \ge \frac{1}{a}.$$

(b) Prove that if f has no root, then

$$\sup_{z\in D} |f(z)| \ge \exp\left(\frac{1-a^2}{4a}\pi\right).$$

(Schweitzer competition, 2012)

Solution→

14.4 Caratheodory's theorem

14.4.1. (10) Is there a Caratheodory type theorem for conformal bijections between domains that are not simply connected and whose boundaries are a union of finitely many Jordan curves?

14.4.2. (9) Show that domains $r_1 < |z| < R_1$ and $r_2 < |z| < R_2$ are biholomorphic if and only if $\dfrac{R_1}{r_1} = \dfrac{R_2}{r_2}$.

14.5 Schwarz reflection principle

14.5.1. (5) Let f be a holomorphic function on $r < |z| < 1$ which extends continuously to the unit circle and satisfies (a) $f(z) \in \mathbb{R}$ for $|z| = 1$ (b) $f \neq 0$, and $|f(z)| = 1$ for $|z| = 1$. Prove that f has an analytic continuation to $r < |z| < \frac{1}{r}$.

14.5.2. (5) Let f be holomorphic and non-vanishing on a convex domain D. Assume that the boundary of D contains the real interval I and that f has a continuous extension to the interior of I where it satisfies $|f| = 1$. Show that f can be analytically continued to $\overline{D} = \{\overline{z} : \ z \in D\}$.

Part II

Solutions

Chapter 15

Hints and final results

1.0.1. Calculate the truth table

$$A \vee (B \Longrightarrow A)$$

Answer:

A	B	A ∨ (B ⇒ A)
I	I	I
I	N	I
N	I	N
N	N	I

←Back

1.0.4. Let $H \subseteq \mathbb{R}$ be a subset. Formalize the following statements and their negations. Is there a set with the given property?

1. H has at most 3 elements.
2. H has no least element.
3. Between any two elements of H there is a third one in H.
4. For any real number there is a greater one in H.

Answer:

1. $\forall \quad x, y, z, w \in H \quad x = y \vee x = z \vee x = w \vee y = z \vee y = w \vee z = w$
2. $\forall \quad x \in H \quad \exists \quad y \in H \quad y < x$
3. $\forall \quad x, y \in H \quad x < y \ \exists \quad z \in H \quad x < z < y$
4. $\forall \quad x \in \mathbb{R} \quad \exists \quad y \in H \quad x < y$

←Back

1.0.8. How many sets $H \subset \{1, 2, \ldots, n\}$ do exist for which $\forall x\ ([(x \in H) \wedge (x+1 \in H)] \Rightarrow x+2 \in H)$?

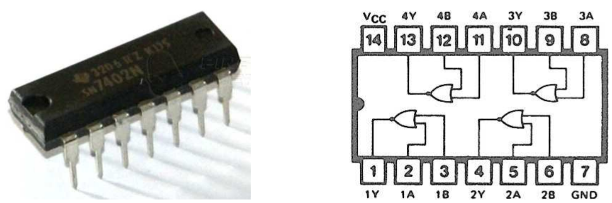
Hint: Add one to the beginning of the set! $j(n+1) = j(n) + j(n-1) + 1$

←Back

1.0.14. Let $\mathrm{NOR}(x, y) = \neg(x \vee y)$. Using only the NOR operation we can create several expressions, e.g., $\mathrm{NOR}(x, \mathrm{NOR}(\mathrm{NOR}(x, y), \mathrm{NOR}(z, x)))$.

(a) Show that we can generate all logic functions of n variables in this way!

(b) Show another example of a logic function of 2-variable NOR with this generating property!



A Texas Instruments SN7402N integrated circuit, with 4 independent NOR logic gates

Hint: It is sufficient to express the operations $\wedge$, $\vee$ and $\neg$.

$$x \wedge y = \mathrm{NOR}(\mathrm{NOR}(x, x), \mathrm{NOR}(y, y); \quad x \vee y = \mathrm{NOR}(\mathrm{NOR}(x, y), \mathrm{NOR}(x, y);$$
$$\neg x = \mathrm{NOR}(x, x).$$

Another "universal" operation is $\mathrm{NAND}(x, y) = \neg(x \wedge y)$. (The integrated circuit SN7400N contains four NAND gates.)

←Back

1.0.22. Prove the so-called *binomial theorem*:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n.$$

Hint: Use exercise 1.0.21 and induction.

←Back

1.0.23. Which one is bigger? 639^9 or $638^9 + 9 \cdot 638^8$?

Hint: Use the binomial theorem.

←Back

1.0.26. Let $A = \{1, 2, ..., n\}$ and $B = \{1, ..., k\}$.

1. How many different functions $f : A \to B$ do exist?
2. How many different injective functions $f : A \to B$ do exist?
3. How many different functions $f : A_0 \to B$ do exist, where $A_0 \subset A$ is arbitrary?

Answer:

1. $|B|^{|A|} = k^n$.
2. $\binom{k}{n} \cdot n! = k(k-1)\cdots(k-n+1)$.
3. $(|B|+1)^{|A|} = (k+1)^n$.

←Back

1.0.32. Is it true for all triples A, B, C of sets that
(a) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$;
(b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$;
(c) $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$?

Answer: (a) yes; (b) yes; (c) no.

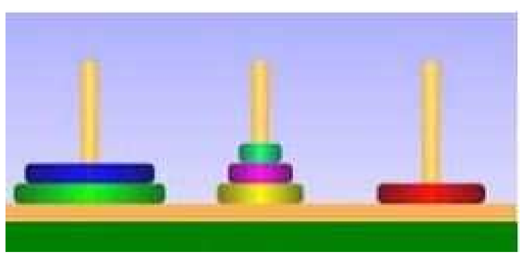
←Back

1.0.44. Prove that $\tan 1^o$ is irrational!

Hint: For which angle do we know that its tangent is irrational?

←Back

1.0.45. At least how many steps do you need to move the 64 stories high Hanoi tower?



Towers of Hanoi

Hint: Induction; $l_{n+1} = 2l_n + 1$.

←Back

1.0.47. For how many parts the space is divided by n planes if no 4 of them have a common point and no 3 of them have a common line?

Hint: Use the result of exercise 1.0.46.

←Back

1.0.55. Prove that the following identity holds for all positive integer n:

$$\sqrt{n} \leq 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Hint: The trivial estimate gives the lower bound, the upper bound can be obtained by induction.

←Back

1.0.68. Let $a, b > 0$. For which x is the expression $\dfrac{a + bx^4}{x^2}$ minimal?

Hint: Apply AM-GM.

←Back

1.1.9. Show that no ordering can make the field of complex numbers into an ordered field.

Hint: Show that $x^2 \geq 0$ holds in every ordered field.

←Back

1.1.12. Does the ordered field of rational functions satisfy the Archimedean axiom?

Hint: The function $x/1$ is greater than all positive integers.

←Back

1.1.13. Given an ordered field R and a subfield $\mathbb{Q}$ show that if

$$(\forall a, b \in R) \Big((1 < a < b < 2) \Rightarrow \Big((\exists q \in \mathbb{Q})\ (a < q < b)\Big)\Big),$$

then R satisfies the Archimedean axiom.

Hint: Suppose that some element $L \in R$ is greater than all positive integers. Let $a = 1 + \frac{1}{2L}$ and $b = 1 + \frac{1}{L}$.

←Back

1.1.14. In which ordered fields can the floor function be defined?

Answer: In Archimedean fields.

←Back

1.1.15. Does the ordered field of rational functions satisfy the Cantor axiom?

Hint: Let $I_n = \left[n; \frac{x}{n}\right]$.

←Back

1.1.18. Which axioms of the reals are satisfied for the set of rational numbers (with the usual operations and ordering)?

Answer: Only the Cantor axiom is not satisfied.

←Back

1.1.37. Does the ordered field of the rational functions satisfy the completeness theorem: all non-empty set has a supremum?

Hint: Consider $\mathbb{R}$ as a subset of the field of the rational functions.

Solution→ ←Back

1.1.38. Prove that if an ordered field satisfies the completeness theorem, then the Archimedean axiom holds.

Hint: What is the supremum of the set of positive integers?

←Back

1.1.39. Prove that if an ordered field satisfies the completeness theorem, then the Cantor axiom holds.

Hint: Suppose that $[a_1, b_1] \supset [a_2, b_2] \supset$ is a descending chain of closed intervals. Show that $\sup\{a_1, a_2, \ldots\}$ is contained by all of the intervals.

←Back

1.1.40. Define recursively the sequence $x_{n+1} = x_n \left(x_n + \frac{1}{n}\right)$ for any x_1. Show that there is exactly one x_1 for which $0 < x_n < x_{n+1} < 1$ for any n.

(IMO 1985/6)

Hint: Let $f_1(x) = x$ and $f_{n+1}(x) = f_n(x)\bigl(f_n(x) + \frac{1}{n}\bigr)$.

(a) For the uniqueness prove that if $x < y$ and the sequences $(f_n(x))$ and $(f_n(y))$ are increasing, then $f_n(y) - f_n(x) > n(y - x)$.

(b) Let a_n and b_n be the real numbers for which $f_{n+1}(a_n) = f_n(a_n)$ and $f_n(b_n) = 1$. Apply Cantor's axiom to the intervals $[a_n, b_n]$.

←Back

2.1.12. Show that every convergent sequence has a minimum or a maximum.

Hint: Show that if the set $A = \{a_n : n \in \mathbb{N}\}$ has no maximum, then the sequence a_n has a subsequence $a_{n_k} \to \sup A$.

←Back

2.1.43. Prove that if $(a_n + b_n)$ is convergent and (b_n) is divergent, then (a_n) is also divergent.

Hint: It is enough to show that if (c_n) is convergent and (d_n) is divergent, then $(c_n + d_n)$ is also divergent.

←Back

2.1.51. Assume that $a_n \to a$ and $a < a_n$ for all n. Prove that a_n can be rearranged to a monotone decreasing sequence.

Hint: Study the sequence $b_n := \max\{a_k : k \geq n\}$.

←Back

2.2.11. Determine the limit of the following recursively defined sequence!
$a_1 = 0,\ a_{n+1} = 1/(1 + a_n)\ (n = 1, 2, \ldots)$.

Hint: See the 2.2.9 exercise.

←Back

2.4.10. Calculate the following:

$$\lim \frac{n^{100}}{1,1^n} = ?$$

Hint: See the solution of 2.2.4.

←Back

2.4.19. Let $a > 0$.

$$\lim \sqrt[n]{n + a^n} = ?$$

Hint: See the solution of 2.4.6.

←Back

2.4.22. Is

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}$$

convergent?

Hint: Check the Cauchy criterion.

←Back

2.8.15. Show that if $|a_{n+1} - a_n| < \frac{1}{n^2}$, then (a_n) is convergent.

Hint: Use the idea of 2.7.2.

←Back

3.2.20. Assume that $g(x) = \lim\limits_{t \to x} f(t)$ exists in every point. Prove that $g(x)$ is continuous.

Hint: f continuous $\Leftrightarrow$ image of convergent sequence is convergent + diagonal method.

←Back

5.3.5. Find the arclength of the curve $r(\theta) = a + a\cos\theta$, $(\theta \in [\pi/4, \pi/4])$.

Hint: Use the formula of arclength in polar coordinates.

←Back

5.4.1. If $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous curve whose image contains $[0,1] \times [0,1]$, can γ be of bounded variation?

Hint: No. Consider a $1/n$-grid on the unit square. For the partition corresponding to the preimages of the vertices of the grid has variation $> n^2 \cdot 1/n$.

←Back

5.4.2. Prove that $f : [0,1] \to \mathbb{R}$ is of bounded variation if and only if it is the sum of two monotonic functions.

Hint: The total variation function minus f is monotone.

←Back

5.5.1. Let f be continuous, $g(x) = \begin{cases} c & \text{if } x < \frac{a+b}{2} \\ d & \text{if } x > \frac{a+b}{2} \\ e & \text{if } x = \frac{a+b}{2} \end{cases}$.

$$\int_a^b f \; dg = ?$$

Hint: $f(\frac{a+b}{2})(d-c)$.

←Back

5.6.6. Is the following integral convergent?

$$\int_0^3 \frac{\cos t}{t} \; \mathrm{d}t$$

Hint: $\frac{\cos t}{t} > \frac{1/2}{t}$. Or: $\frac{\cos t}{t} > \frac{1 - t^2/2}{t}$. Or: Integration by parts $1/t = u', \cos t = v$ leads to a proper integral.

←Back

6.0.31. Convergent or divergent?

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Hint: Use the 6.0.30 condensation lemma.

←Back

6.0.32. Let $\varepsilon > 0$. Convergent or divergent?

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}}$$

Hint: Use the 6.0.30 condensation lemma.

←Back

8.1.31. $\lim_{(0,0)}(x^2+y^2)^{x^2y^2}=?$

Answer: 1

←Back

10.2.3. For what functions $f:\mathbb{R}^2\to\mathbb{R}$ will the following statement be true? If g is a simple, closed rectifiable curve in $\mathbb{R}^2$, then

$$\int_g x^2y^3\,\mathrm{d}y=\int_g f(x,y)\,\mathrm{d}x.$$

Answer: $f(x,y)=-\frac{1}{2}xy^4+c(x)$ with some differentiable function $c(x)$.

←Back

10.3.11. Is $H=\mathbb{R}^3\setminus\{(\cos t,\sin t,e^t):t\in\mathbb{R}\}$ simply connected?

Answer: Yes.

←Back

11.1.2. What is the smallest possible cardinality of an infinite σ-ring?

Answer: Continuum.

←Back

11.6.3. True or false? If f is absolutely continuous and strictly increasing on $[a,b]$, then its inverse is also absolutely continuous.

Answer: No.

←Back

12.0.1.

$$\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots=?$$

Hint: Expand $(1+x)^n$ by the binomial theorem.

←Back

12.0.2. Let $a,b,c\in\mathbb{C}$. What is the geometric interpretation of

$$\frac{1}{2}\,\mathrm{Im}\left((c-a)\cdot\overline{(b-a)}\right)?$$

Answer: The signed area of the triangle (a, b, c).

←Back

12.0.4. What are the product, the sum and the sum of squares of the complex mth roots of unity?

Hint: Use the fact that these are exactly the roots of $x^m - 1$.

←Back

12.0.10. Let $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ be the so-called Zhukowksy map. What is the image of
(a) the unit circle?
(b) the interior of the unit circle?
(c) the exterior of the unit circle?
(d) the circles with center 0?
(e) the lines passing through 0?

Answer: (a): The line segment $[-1, 1]$.
(b) and (c): The complement of $[-1, 1]$.
(d): Ellipses with foci $-1, 1$. (The unit circle is mapped to the line segment $[-1, 1]$.)
(e): Hyperbolas with foci $-1, 1$. (The image of the real axis is the union of the rays $(-\infty, -1]$ and $[1, \infty)$; the imaginary axis is mapped onto itself.)

←Back

13.1.7. Let $a, b \in \mathbb{C}$ and $|b| < 1$. Prove that

$$\frac{1}{2\pi} \int_{|z|=1} \left| \frac{z-a}{z-b} \right|^2 |\,\mathrm{d}z| = \frac{|a-b|^2}{1-|b|^2} + 1.$$

Hint: Transform it to a contour integral, then apply Cauchy's formula.

Solution→ ←Back

13.1.9. The function $f(z)$ is holomorphic in the interior of the unit disc ($|z| < 1$) and $|f| < 1$. How large can $|f'''(0)|$ be?

Answer: 6.

←Back

13.3.1. An entire function $f(z)$ satisfies $|f(1/n)| = 1/n^2$ for $n = 1, 2, \ldots$, and $|f(i)| = 2$. What are the possible values of $|f(-i)|$?

Hint: Apply the Unicity Theorem to $g(z) = f(z) \cdot \overline{f(\bar{z})}$.

Solution→ ←Back

13.3.2. Show that if f takes only real values on the real and imaginary axes, then f is even.

Hint: Consider the entire functions $\overline{f(\overline{z})}$ and $\overline{f(-\overline{z})}$.

←Back

Chapter 16

Solutions

1.0.12. Prove that the implication is left distributive with respect to the disjunction.

Solution: We have to prove

$$\Big(A \Rightarrow (B \vee C)\Big) = (A \Rightarrow B) \vee (A \Rightarrow C).$$

By the basic properties of the $\vee$ operation (idempotency, commutativity, associativity) and the identity $(X \Rightarrow Y) \Rightarrow \neg X \vee Y$,

$$\begin{aligned}\Big(A \Rightarrow (B \vee C)\Big) &= \neg A \vee (B \vee C) = (\neg A \vee \neg A) \vee (B \vee C) \\ &= (\neg A \vee B) \vee (\neg A \vee C) = (A \Rightarrow B) \vee (A \Rightarrow C).\end{aligned}$$

←Back

1.0.42. Prove that

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\dots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Solution: Induction: The statement is true for $n = 1$, and

$$a_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) a_n,$$

assuming that the statement is true for a_n, we get

$$a_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \frac{n+1}{2n} = \frac{n+2}{2n+2}.$$

←Back

1.0.49. Prove that the following identity holds for all positive integer n:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}.$$

Solution: Induction on n. For $n=1$ we have $\frac{1}{1\cdot 3}=\frac{1}{3}\,\surd$. Suppose now that the identity holds for n, then for $n+1$ we have

$$\begin{aligned} L.H.S. &= \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)\cdot(2n+1)}+\frac{1}{(2n+1)\cdot(2n+3)} \\ &= \frac{n}{2n+1}+\frac{1}{(2n+1)\cdot(2n+3)} \qquad \text{by the ind. hyp.} \\ &= \frac{n(2n+3)+1}{(2n+1)\cdot(2n+3)}=\frac{2n^2+3n+1}{(2n+1)\cdot(2n+3)}=\frac{n+1}{2(n+1)+1}, \end{aligned}$$

since $2n^2+5n+1=(2n+1)(n+1)$.

Solution 2: Since $\frac{1}{(2n-1)\cdot(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$, we get a telescopic sum, therefore

$$\begin{aligned} 2\cdot L.H.S. &= \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) \\ &= 1-\frac{1}{2n+1}=\frac{2n}{2n+1}. \end{aligned}$$

←Back

1.0.51. Prove that the following identity holds for all positive integer n:

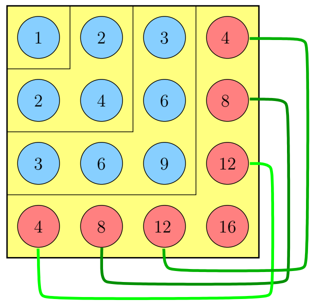
$$1^3+\ldots+n^3=\left(\frac{n\cdot(n+1)}{2}\right)^2.$$

Solution: Induction on n. For $n=1$ both sides equal to 1. If the statement holds for n, then for $n+1$ we have

$$1^3+\ldots+n^3+(n+1)^3=\left(\frac{n\cdot(n+1)}{2}\right)^2+(n+1)^3=$$

$$=(n+1)^2\left(\frac{n^2}{4}+n+1\right)=(n+1)^2\frac{(n+2)^2}{4}=\left(\frac{(n+1)(n+2)}{4}\right)^2.$$

Solution 2.

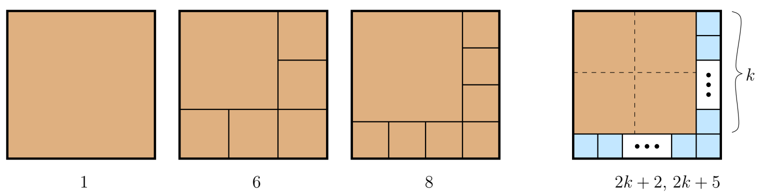


The sum of the numbers in the n-th square is $(\sum i)^2$, the sum of the numbers connected with curves is n^2, and we have $n-1$ on one level and we also have n^2 in the lower right corner.

←Back

1.0.56. Show that for all positive integer $n \geq 6$ a square can be divided into n squares.

Solution: Dividing a square into for ones of half the side we see that if a square can be divided into n squares, then it can also be divided into $n+3$ squares. On the other hand we have the solutions for $1, 6$ and 8:



(The right-most picture shows another possible construction.)

←Back

1.0.66. Prove that if $a, b, c > 0$, then the following inequality holds

$$\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} \geq 3.$$

Solution: Apply the AM-GM inequality to the terms on the left-hand side:

$$\frac{\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab}}{3} \geq \sqrt[3]{\frac{a^2}{bc} \cdot \frac{b^2}{ac} \cdot \frac{c^2}{ab}} = \sqrt[3]{1} = 1.$$

←Back

1.0.74. Which rectangular box has the greatest volume among the ones with given surface area?

Solution: $A = 2(ab+ac+bc) = 6\frac{ab+ac+bc}{3} \overset{\text{sz-m}}{\geq} 6\sqrt[3]{a^2b^2c^2} = 6V^{2/3}$. Equality can occur only for $ab = ac = bc$, i.e. for the case of the cube.

←Back

1.0.77. Calculate the maximum value of the function $x^2 \cdot (1-x)$ for $x \in [0,1]$.

Solution: By the AM-GM inequality,

$$\sqrt[3]{x \cdot x \cdot (2-2x)} \overset{\text{AM-GM}}{\leq} \frac{x+x+(2-2x)}{3}.$$

←Back

1.0.78. Prove that the cylinder with the least surface area among the ones with given volume V is the cylinder whose height equals the diameter of its base.

Solution: $\dfrac{A}{3\pi} = \dfrac{2r^2+rh+rh}{3} \overset{\text{AM-GM}}{\geq} \sqrt[3]{2r^2 \cdot rh \cdot rh} = \sqrt[3]{2\dfrac{V^2}{\pi^2}}.$

←Back

1.0.79. Prove that $n! < \left(\dfrac{n+1}{2}\right)^n$.

Solution: $\sqrt[n]{n!} \overset{\text{AM-GM}}{\leq} \dfrac{\binom{n+1}{2}}{n}$ for $n > 1$.

←Back

1.0.83. Prove that for any sequence $a_1, a_2, \ldots, a_n$ of positive real numbers,

$$\frac{1}{\frac{1}{a_1}} + \frac{2}{\frac{1}{a_1}+\frac{1}{a_2}} + \frac{3}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}} + \ldots + \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}} < 2(a_1+a_2+\ldots+a_n).$$

(KöMaL N. 189., November 1998)

Solution: Applying the weighted AM-HM inequality,

$$\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_k}} = \sum_{k=1}^{n} \frac{2}{k+1} \cdot \frac{1+2+\ldots+k}{\frac{1}{a_1}+\frac{2}{2a_2}+\ldots+\frac{k}{ka_k}} \leq$$

$$\leq \sum_{k=1}^{n} \frac{2}{k+1} \cdot \frac{1 \cdot a_1 + 2 \cdot 2a_2 + \ldots + k \cdot ka_k}{1+2+\ldots+k} =$$

$$= \sum_{k=1}^{n} \frac{4}{k(k+1)^2} \sum_{i=1}^{k} i^2 a_i = \sum_{i=1}^{n} i^2 a_i \sum_{i=k}^{n} \frac{4}{k(k+1)^2} < \sum_{i=1}^{n} i^2 a_i \sum_{i=k}^{n} \frac{2(2k+1)}{k^2(k+1)^2} =$$

$$= \sum_{i=1}^{n} i^2 a_i \sum_{i=k}^{n} \left(\frac{2}{k^2} - \frac{2}{(k+1)^2}\right) < \sum_{i=1}^{n} i^2 a_i \left(\frac{2}{i^2} - \frac{2}{(n+1)^2}\right) <$$

$$< \sum_{i=1}^{n} i^2 a_i \cdot \frac{2}{i^2} = 2\sum_{i=1}^{n} a_i.$$

Remark: The constant 2 on the right-hand side is sharp. If $a_i = \frac{1}{i}$ and n is sufficiently large, the ratio between the two sides can be arbitrarily close to 1.

←Back

1.1.3. Using the field axioms prove the following statement: $(-a)(-b) = ab$.

Solution: $a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a = (1 + (-1)) \cdot a = 0$, because of the definition of 1 and -1 and distributivity. Therefore the uniqueness of the additive inverse implies $(-1) \cdot a = -a$. $\implies$ $(-a)(-b) = ((-1) \cdot a)((-1) \cdot b)$, which further equals $((-1) \cdot (-1))ab$ because associativity of multiplication and commutativity. Finally it is easy to see that $(-1) \cdot (-1) = 1$.

←Back

1.1.37. Does the ordered field of the rational functions satisfy the completeness theorem: all non-empty set has a supremum?

Solution: No.

Denote by $\mathbb{R}(x)$ the ordered field of the rational functions. Mapping the real numbers to the constant functions, $\mathbb{R}$ can be considered as an ordered subfield of $\mathbb{R}(x)$. We show that $\mathbb{R}$ is non-empty, bounded from above but it has no smallest upper bound.

$\mathbb{R}$ is obviously non-empty. The function $x = \dfrac{x}{1} \in \mathbb{R}(x)$ is an upper bound of $\mathbb{R}$ because for any $a \in \mathbb{R}$ we have $x - a = \dfrac{x-a}{1} > 0$. Hence, $\mathbb{R}$ is a non-empty subset of $\mathbb{R}(x)$ and it is bounded from above.

Now we show that $\mathbb{R}$ has no smallest upper bound. If $K \in \mathbb{R}(x)$ is an upper bound, then $K - 1$ is also an upper bound since for every $a \in \mathbb{R}$ we have $a + 1 \in \mathbb{R} \Rightarrow a + 1 \le K \Rightarrow a \le K$.

←Back

1.1.42. Prove that $(1 + x)^r \le 1 + rx$ if $r \in \mathbb{Q}$, $0 < r < 1$ and $x \ge -1$.

Solution: $r = p/q$, $\sqrt[q]{(1+x)^p \cdot 1^{q-p}} \overset{\text{AM-GM}}{\le} \frac{p(1+x)+(q-p)}{q}$.

←Back

2.1.18. Is it true that if x_n is convergent, y_n is divergent, then $x_n y_n$ is divergent?

Solution: No, for example $x_n = \frac{1}{n^2}$ and $y_n = n$.

←Back

2.1.27. Is there a sequence of irrational numbers converging to (a) 1, (b) $\sqrt{2}$?

Solution: (a) $1 + \frac{\sqrt{2}}{n}$ (b) $\left(1 + \frac{1}{n}\right)\sqrt{2}$.

←Back

2.1.30. Does $a_n^2 \to a^2$ imply that $a_n \to a$? And does $a_n^3 \to a^3$ imply that $a_n \to a$?

Solution: $(-1)^n \not\to 1$. But for $a = 0$ we have $\delta_{a_n}(\varepsilon) := \delta_{a_n^3}(\varepsilon^3)$ if $a > 0$, then $|a_n - a| = \frac{|a_n^3 - a^3|}{a_n^2 + aa_n + a^2} \leq \frac{|a_n^3 - a^3|}{3(a/2)^2}$ for n big enough.

←Back

2.1.47. Let $a_k \neq 0$ and $p(x) = a_0 + a_1 x + \ldots + a_k x^k$. Prove that

$$\lim_{n\to+\infty} \frac{p(n+1)}{p(n)} = 1.$$

Solution: Simplify by $a_0 n^k$:

$$\frac{p(n+1)}{p(n)} = \frac{\left(1 + \frac{1}{n}\right)^k + a(n)}{1 + b(n)},$$

where $a(n) \to 0$ and $b(n) \to 0$.

←Back

2.1.54. Prove that if the sequence (a_n) has no convergent subsequence, then $|a_n| \to \infty$.

Solution: If the sequence $|a_n| \not\to \infty$, then it has a bounded subsequence. By the Bolzano–Weierstrass theorem this subsequence has a convergent subsequence.

←Back

2.2.2. Prove that $n^{n+1} > (n+1)^n$ if $n > 2$.

Solution: Consider the inequality between the arithmetic and geometric means for the numbers $\overbrace{n+1, \ldots, n+1}^{n-1}, \sqrt{n+1}, \sqrt{n+1}$.

←Back

2.2.3. Prove that

$$\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \ldots \cdot \sqrt[2^n]{2^n} < n+1.$$

Solution: $a_n = 2^{b_n}$, where $b_n = \frac{1}{2} + \frac{2}{4} + \cdots + \frac{n}{2^n}$. It is easy to check by induction that $2 - b_n = \frac{n+2}{2^n}$, therefore $a_n < 4$.

←Back

2.2.4. Prove that $2^n > n^k$ holds for all sufficiently (depending on k) large n.

Solution: $2^n > \binom{n}{k+1}$ if $n > k+1$. $\binom{n}{k+1} > \frac{1}{(k+1)!}(n/2)^{k+1}$ if $n > 2(k+1)$. $\frac{1}{(k+1)!}(n/2)^{k+1} > n^k$ if $n > 2^{k+1}(k+1)!$. This estimate is not sharp: $\frac{n}{\log_2 n} > k$. E.g. for $k = 10$ it holds from $n = 60$.

←Back

2.2.10. Prove that for the sequence $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{a_n}$ we have $a_{10001} > 100$ (see the 2.2.9 exercise and its solution.)

Solution: a_n is monotone icreasing. Assume that $a_{n^2+1} < n \Rightarrow \frac{1}{a_i} > \frac{1}{n}\ \forall i \leq n \Rightarrow a_{n^2+1} > a_1 + n^2 \frac{1}{n}$ ↯

←Back

2.3.1. Find a non-convergent sequence with exactly one limit point.

Solution: Merge the sequences $1/n$ and n.

←Back

2.3.5. Find a sequence such that the set of limit points of it is $[0, 1]$.

Solution: List the elements of a countable dense subset of $[0, 1]$. (E.g. $[0, 1] \cap \mathbb{Q}$.)

←Back

2.4.6. Calculate $\lim_{n\to\infty} \sqrt[n]{2^n - n}$.

Solution:

$$2 = \sqrt[n]{2^n} > \sqrt[n]{2^n - n} > \sqrt[n]{2^n - 2^{n-1}} = 2\sqrt[n]{\frac{1}{2}},$$

for n big enough. The RHS tends to 2 by 2.4.5, so the sandwich theorem implies the result.

←Back

2.4.17.

$$\lim \frac{1}{n(\sqrt{n^2-1} - n)} =?$$

Solution:

$$\frac{1}{n(\sqrt{n^2-1} - n)} = \frac{1}{n(\sqrt{n^2-1} - n)} \frac{\sqrt{n^2-1} - n}{\sqrt{n^2-1} - n} = \frac{\sqrt{1 - \frac{1}{n^2}} + 1}{-1},$$

therefore $\lim \frac{1}{n(\sqrt{n^2-1}-n)} = -2$.

←Back

2.4.24. Is
$$\sqrt[n]{n^2+\cos n}$$
convergent?

Solution: $1 < \sqrt[n]{n^2+\cos n} < \sqrt[n]{n^3} = \left(\sqrt[n]{n}\right)^3 \to 1^3 = 1.$

←Back

2.5.19. Let $a_1 = 1$, $a_{n+1} = a_n + \dfrac{2}{a_n^2}$. Prove the existence of an $n \in \mathbb{N}$, for which $a_n \geq 10$.

Solution: Suppose that $\forall n\ a_n < 10. \implies a_n^2 < 100 \implies \frac{2}{a_n^2} > \frac{2}{100} \implies a_{n+1} = a_n + \frac{2}{a_n^2} > a_n + \frac{2}{100}$. by induction we get
$$a_{n+1} > a_1 + n \cdot \frac{2}{100} = 1 + n \cdot \frac{2}{100},$$
consequently for e.g.
$$n = 500\ a_{501} > 1 + 500 \cdot \frac{2}{100} = 11,$$
which contradicts to our assumption.

←Back

2.6.4. Prove that
$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2},$$
in other words the sequence $a_n = \left(1+\frac{1}{n}\right)^{n+1}$ is strictly monotone decreasing.

Solution: equivalently $$\sqrt[n+2]{\left(\frac{n}{n+1}\right)^{n+1} \cdot 1} \overset{\text{a-g}}{<} \frac{(n+1)\left(\frac{n}{n+1}\right)+1}{n+2}.$$

←Back

2.6.10. Calculate the limit of the sequence
$$a_n = \left(\frac{n+2}{n+1}\right)^n.$$

Solution:
$$a_n\left(1+\frac{1}{n+1}\right) = \left(1+\frac{1}{n+1}\right)^{n+1} \to e,$$
therefore $a_n \to e$.

←Back

2.7.1. The sequence a_n is monotone and it has a convergent subsequence. Does it imply that a_n is convergent?

Solution: Yes, since we have an $a_{n_k} \to a$ convergent subsequence and because of the monotonicity $\forall\ n > n_k\ |a_n - a| \leq |a_{n_k} - a|$, therefore $a_n \to a$.

←Back

2.8.6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

Solution: $\frac{1}{n^2} < \frac{1}{(n-1)n}$ and $\sum_{n=2}^{\infty} \frac{1}{(n-1)n} = 1$ (telescopic sum).

←Back

2.8.8. Find a sequence a_n such that $\sum a_n$ is convergent, and a_{n+1}/a_n is not bounded.

Solution: For example $a_{2n} = \frac{1}{n^2}$ and $a_{2n+1} = \frac{1}{n^3}$.

←Back

3.1.2. Show that the following functions are injective on the given set H, and calculate the inverse.

1. $f(x) = \dfrac{x}{x+1}$, $H = [-1, 1]$; 2. $f(x) = \dfrac{x}{|x|+1}$, $H = \mathbb{R}$.

Solution: $f^{-1}(y) = \frac{y}{1-|y|}$, $y \in (-1, 1)$.

←Back

3.1.6. Are the following functions injective on $[-1, 1]$?

a) $f(x) = \dfrac{x}{x^2+1}$, b) $g(x) = \dfrac{x^2}{x^2+1}$.

Solution: a) Let $x \neq y$ and suppose that $f(x) = f(y)$, i.e.,

$$\frac{x}{x^2+1} = \frac{y}{y^2+1} \implies x(y^2+1) = y(x^2+1) \implies x-y = (x-y)xy \implies 1 = xy,$$

since $x - y \neq 0$. On the other hand $|x|, |y| \leq 1$, which can be satisfied only for $x = y = \pm 1$ but equality was not allowed. Therefore $f(x)$ is injective on $[-1, 1]$.

b) $g(1) = g(-1)$, therefore $g(x)$ is not injective on $[-1, 1]$.

←Back

3.4.2. (Brouwer fixed-point theorem; 1-dimensional case.) All $f : [a, b] \to [a, b]$ continuous functions have a fixed point, i.e., an x, for which $f(x) = x$.

Solution: Apply the Bolzano–Darboux theorem to $f(x) - x$.

←Back

3.4.7. Prove that the polynomial $x^3 - 3x^2 - x + 2$ has 3 real roots.

Solution: $f(-1) = -1$, $f(0) = 2$, $f(2) = -4$, $f(4) = 14$. By the Bolzano–Darboux theorem there are at least 3 real roots.

←Back

4.4.3. Let $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$ be real numbers. Show that

$$\det \begin{pmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_2 b_1} & e^{a_2 b_2} & \ldots & e^{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_n b_1} & e^{a_n b_2} & \ldots & e^{a_n b_n} \end{pmatrix} > 0.$$

(KöMaL A. 463., October 2008)

Solution: Apply induction on n. For $n = 1$ the statement is $e^{a_1 b_1} > 0$ which is obvious. Now suppose $n > 1$ and assume that the statement is true for all smaller values.

Let $c_i = a_i - a_1 > 0$. Then

$$\det \begin{pmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_2 b_1} & e^{a_2 b_2} & \ldots & e^{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_n b_1} & e^{a_n b_2} & \ldots & e^{a_n b_n} \end{pmatrix} =$$

$$= \det \begin{pmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_1 b_1} e^{c_2 b_1} & e^{a_1 b_2} e^{c_2 b_2} & \ldots & e^{a_1 b_n} e^{c_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_1 b_1} e^{c_n b_1} & e^{a_1 b_2} e^{c_n b_2} & \ldots & e^{a_1 b_n} e^{c_n b_n} \end{pmatrix} =$$

$$= e^{a_1(b_1 + b_2 + \cdots + b_n)} \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ e^{c_2 b_1} & e^{c_2 b_2} & \ldots & e^{c_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_1} & e^{c_n b_2} & \ldots & e^{c_n b_n} \end{pmatrix},$$

so it is sufficient to prove that the last determinant is positive.

To eliminate the first row, subtract the $(n-1)$th column from the nth column. Then subtract the $(n-2)$th column from the $(n-1)$th column, and so on, finally subtract the first column from the second column. Then

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ e^{c_2b_1} & e^{c_2b_2} & \dots & e^{c_2b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_1} & e^{c_nb_2} & \dots & e^{c_nb_n} \end{pmatrix} =$$

$$= \det\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ e^{c_2b_1} & e^{c_2b_2}-e^{c_2b_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3b_1} & e^{c_3b_2}-e^{c_3b_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_1} & e^{c_nb_2}-e^{c_nb_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix} =$$

$$= \det\begin{pmatrix} e^{c_2b_2}-e^{c_2b_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3b_2}-e^{c_3b_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_2}-e^{c_nb_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix}.$$

Consider the function

$$f(t) = \det\begin{pmatrix} e^{c_2t} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3t} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nt} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix}.$$

Then

$$\det\begin{pmatrix} e^{c_2b_2}-e^{c_2b_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3b_2}-e^{c_3b_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_2}-e^{c_nb_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix} = f(b_2)-f(b_1).$$

By Lagrange's mean value theorem, there exists a $b_1 < x_1 < b_2$ such that $f(b_2) - f(b_1) = (b_2 - b_1)f'(x_1)$, i.e.,

$$\det\begin{pmatrix} e^{c_2b_2}-e^{c_2b_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3b_2}-e^{c_3b_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_2}-e^{c_nb_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix} =$$

$$= (b_2 - b_1)\det\begin{pmatrix} c_2e^{c_2x_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ c_3e^{c_3x_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ c_ne^{c_nx_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix}.$$

Repeating the same argument for each column, it can be obtained that there exist real numbers $x_i \in (b_i, b_{i+1})$ $(1 \le i \le n-1)$ such that

$$\det\begin{pmatrix} e^{c_2b_2}-e^{c_2b_1} & e^{c_2b_3}-e^{c_2b_2} & \dots & e^{c_2b_n}-e^{c_2b_{n-1}} \\ e^{c_3b_2}-e^{c_3b_1} & e^{c_3b_3}-e^{c_3b_2} & \dots & e^{c_3b_n}-e^{c_3b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nb_2}-e^{c_nb_1} & e^{c_nb_3}-e^{c_nb_2} & \dots & e^{c_nb_n}-e^{c_nb_{n-1}} \end{pmatrix} =$$

$$= \prod_{i=1}^{n-1}(b_{i+1}-b_i)\cdot\det\begin{pmatrix} c_2e^{c_2x_1} & c_2e^{c_2x_2} & \dots & c_2e^{c_2x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ c_ne^{c_nx_1} & c_ne^{c_nx_2} & \dots & c_ne^{c_nx_{n-1}} \end{pmatrix} =$$

$$= \prod_{i=1}^{n-1}(b_{i+1}-b_i)\cdot\prod_{i=2}^{n}c_i\cdot\det\begin{pmatrix} e^{c_2x_1} & e^{c_2x_2} & \dots & e^{c_2x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_nx_1} & e^{c_nx_2} & \dots & e^{c_nx_{n-1}} \end{pmatrix}.$$

By the induction hypothesis, this is positive.

←Back

4.5.15. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be a polynomial with real coefficients and $n \ge 2$, and suppose that the polynomial $(x-1)^{k+1}$ divides $p(x)$ with some positive integer k. Prove that

$$\sum_{\ell=0}^{n-1}|a_\ell| > 1 + \frac{2k^2}{n}.$$

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Solution: For convenience, define the leading coefficient $a_n = 1$ also.

Lemma 1. *For every polynomial $q(y)$ with degree at most k, we have* $\sum\limits_{\ell=0}^{n} a_\ell\, q(\ell) = 0.$

Proof. Let $\varphi_0(y) = 1$ and let $\varphi_\nu(y) = y(y-1)\ldots(y-\nu+1)$ for $\nu = 1, 2, \ldots$. By $(x-1)^k \big| p(x)$, for $0 \le \nu \le k$ we have

$$\sum_{\ell=0}^{n} a_\ell\, \varphi_\nu(\ell) = f^{(\nu)}(1) = 0.$$

The polynomials $\varphi_0(y),\dots,\varphi_k(y)$ form a basis of the vector space of polynomials with degree at most k, so $q(y)=\sum\limits_{\nu=0}^{k} c_\nu\varphi_\nu(y)$ with some real numbers $c_0,\dots,c_k$. Then

$$\sum_{\ell=0}^{n} a_\ell\, q(\ell)=\sum_{\ell=0}^{n} a_\ell\left(\sum_{\nu=0}^{k} c_\nu\,\varphi_\nu(\ell)\right)=\sum_{\nu=0}^{k} c_\nu\left(\sum_{\ell=0}^{n} a_\ell\,\varphi_\nu(\ell)\right)=0. \qquad \square$$

$\square$

To prove the problem statement, let T_k be the kth Chebyshev polynomial, and choose

$$q(y)=T_k\left(\frac{2}{n-1}y-1\right).$$

Then $q(0),\dots,q(n-1)\in T_k\big([-1,1]\big)=[-1,1]$, and

$$\begin{aligned} q(n)=T_k\left(\frac{n+1}{n-1}\right)&=\cosh\left(k\cdot\cosh^{-1}\frac{n+1}{n-1}\right)=\\ &=\cosh\left(k\cdot\log\left(\tfrac{n+1}{n-1}+\sqrt{\left(\tfrac{n+1}{n-1}\right)^2-1}\right)\right)\\ &=\cosh\left(k\cdot\log\frac{(\sqrt{n}+1)^2}{n-1}\right)=\cosh\left(k\cdot\log\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)>\cosh\frac{2k}{\sqrt{n}}. \end{aligned}$$

(In the last step we applied the inequality $\log\frac{1+x}{1-x}>2x$.)

By applying the lemma,

$$\sum_{\ell=0}^{n-1}|a_\ell|\ge\sum_{\ell=0}^{n-1}a_\ell\big(-q(\ell)\big)=q(n)>\cosh\frac{2k}{\sqrt{n}}>1+\frac{2k^2}{n}.$$

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6.0.30. Prove the *Condensation lemma*: Let $a_1\ge a_2\ge\cdots\ge a_n\ge\cdots\ge 0$. Then

$$\sum_{n=1}^{\infty}a_n \qquad \text{convergent} \iff \sum_{k=1}^{\infty}2^k a_{2^k} \qquad \text{convergent.}$$

Solution:

$$\begin{array}{cccccccccc} a_1+ & a_2+ & a_2+ & a_4+ & a_4+ & a_4+ & a_4+ & a_8+ & \cdots & \ge \\ a_1+ & a_2+ & a_3+ & a_4+ & a_5+ & a_6+ & a_7+ & a_8+ & \cdots & \ge \\ \frac{1}{2}a_1+ & a_2+ & a_4+ & a_4+ & a_8+ & a_8+ & a_8+ & a_8+ & \cdots & \end{array}$$

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11.1.6. Prove that if $f : \mathbb{R} \to \mathbb{R}$, then the set of points of continuity is Borel, and give as small as possible of Borel class (e.g. $G_{\delta\sigma\delta\sigma\delta\sigma\delta\sigma}$), to which it still belongs.

Solution: For every positive integer n let

$$\mathcal{I}_n = \left\{ I \subset \mathbb{R} : \ I \text{ is an open interval and } \sup_I f - \inf_I f < \tfrac{1}{n} \right\}$$

and let

$$A_n = \cup \mathcal{I}_n = \bigcup_{I \in \mathcal{I}_n} I.$$

By Cauchy's criterion, any $a \in \mathbb{R}$ is a point of continuity of f if and only if

$$\forall n \in \mathbb{N} \ \ \exists I \in \mathcal{I}_n \ \ a \in I,$$

or equivalently

$$\forall n \in \mathbb{N} \ \ a \in A_n.$$

Therefore, the set of points of continuity is $\bigcap_{n \in \mathbb{N}} A_n$, that is in G_δ.

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12.0.9. Let $n \geq 2$ and $u_1 = 1, u_2, \dots, u_n$ be complex numbers with absolute value at most 1, and let

$$f(z) = (z - u_1)(z - u_2) \dots (z - u_n).$$

Show that the polynomial $f'(z)$ has a root with non-negative real part.

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Solution: If 1 is a multiple root of f, then $f'(1) = 0$ and the statement becomes trivial. So we assume that $u_2, \dots, u_n \neq 1$.

Let the roots of $f'(z)$ be $v_1, v_2 \dots, v_{n-1}$, and consider the polynomial $g(z) = f(1 - z) = a_1 z + a_2 z^2 + \dots + a_n z^n$.

The non-zero roots of $g(z)$ are $1 - u_2, \dots, 1 - u_n$. From the Viéta formulas we obtain

$$\sum_{k=2}^{n} \frac{1}{1 - u_k} = \frac{(1 - u_2) \dots (1 - u_{n-1}) + \dots + (1 - u_3) \dots (1 - u_n)}{(1 - u_2) \dots (1 - u_n)} = -\frac{a_2}{a_1}.$$

The roots of the polynomial $f'(1 - z) = -g'(z) = -a_1 - 2a_2 z - \dots - n a_n z^{n-1}$ are $1 - v_1, \dots, 1 - v_{n-1}$; from the Viéta formulas again,

$$\sum_{\ell=1}^{n-1} \frac{1}{1 - v_\ell} = \frac{(1 - v_1) \dots (1 - v_{n-2}) + \dots + (1 - v_2 \dots v_{n-1})}{(1 - v_1) \dots (1 - v_{n-1})} = -\frac{2a_2}{a_1}.$$

Combining the two equations,

$$\sum_{\ell=1}^{n-1} \frac{1}{1-v_\ell} = 2\sum_{k=2}^{n} \frac{1}{1-u_k}.$$

For every k, the number u_k lies in the unit disc (or on its boundary), and $1-u_k$ lies in the circle with center 1 and unit radius (or on its boundary). The operation of taking reciprocals can be considered as the combination of an inversion from pole 0 and mirroring over the real axis. Hence $\frac{1}{1-u_k}$ lies in the half plane $\mathrm{Re}\, z \ge \frac{1}{2}$, i.e. $\mathrm{Re}\frac{1}{1-u_k} \ge \frac{1}{2}$.

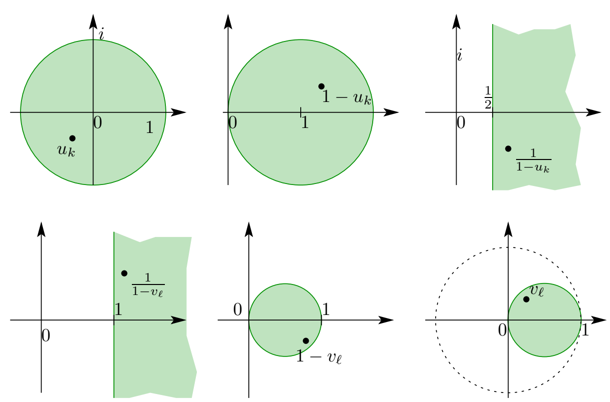
Summing up these inequalities,

$$\max_{1\le \ell\le n-1} \mathrm{Re}\frac{1}{1-v_\ell} \ge \frac{1}{n-1}\sum_{\ell=1}^{n-1}\mathrm{Re}\frac{1}{1-v_\ell} = \frac{2}{n-1}\sum_{k=2}^{n}\mathrm{Re}\frac{1}{1-u_k} \ge 1,$$

so at least one $\frac{1}{1-v_\ell}$ lies in the half plane $\mathrm{Re}\, z \ge 1$.

Repeating the same geometric steps backwards,

$$\mathrm{Re}\frac{1}{1-v_\ell} \ge 1 \iff \left|(1-v_\ell)-\frac{1}{2}\right| \le \frac{1}{2} \iff \left|v_\ell - \frac{1}{2}\right| \le \frac{1}{2} \implies \mathrm{Re}\, v_\ell \ge 0.$$



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13.1.7. Let $a, b \in \mathbb{C}$ and $|b| < 1$. Prove that

$$\frac{1}{2\pi}\int_{|z|=1} \left|\frac{z-a}{z-b}\right|^2 |\,\mathrm{d}z| = \frac{|a-b|^2}{1-|b|^2} + 1.$$

Solution:

$$\frac{1}{2\pi}\int_{|z|=1}\left|\frac{z-a}{z-b}\right|^2 |\,\mathrm{d}z| = \frac{1}{2\pi}\int_{|z|=1}\frac{(z-a)(\overline{z}-\overline{a})}{(z-b)(\overline{z}-\overline{b})}\cdot\frac{\mathrm{d}z}{iz} =$$
$$= \frac{1}{2\pi i}\int_{|z|=1}\frac{(z-a)(\frac{1}{z}-\overline{a})}{(z-b)(\frac{1}{z}-\overline{b})}\cdot\frac{\mathrm{d}z}{z} =$$
$$= \frac{1}{2\pi i}\int_{|z|=1}\frac{(z-a)(1-\overline{a}z)}{b(1-\overline{b}z)}\left(\frac{1}{z-b}-\frac{1}{z}\right)\mathrm{d}z =$$
$$= \left.\frac{(z-a)(1-\overline{a}z)}{b(1-\overline{b}z)}\right|_{z=b} - \left.\frac{(z-a)(1-\overline{a}z)}{b(1-\overline{b}z)}\right|_{z=0} =$$
$$= \frac{(b-a)(1-\overline{a}b)}{b(1-\overline{b}b)} + \frac{a}{b} = \frac{(a-\overline{b})(\overline{a}-b)}{1-b\overline{b}} + 1 = \frac{|a-b|^2}{1-|b|^2} + 1.$$

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13.3.1. An entire function $f(z)$ satisfies $|f(1/n)| = 1/n^2$ for $n = 1, 2, \ldots$, and $|f(i)| = 2$. What are the possible values of $|f(-i)|$?

Solution: Let $g(z) = f(z)\cdot\overline{f(\overline{z})}$, which also is an entire function. At the points of the form $1/n$ we have $g(1/n) = f(1/n)\cdot\overline{f(1/n)} = |f(1/n)|^2 = (1/n)^4$. Hence, by the Unicity Theorem, $g(z) = z^4$. Then $1 = |i^4| = |g(i)| = |f(i)|\cdot|f(-i)| = 2|f(-i)|$, so $|f(-i)| = \frac{1}{2}$.

Remark: The property $|g(1/n)| = 1/n^2$ is satisfied by the functions of the form $f(z) = z^2 e^{i\varphi(z)}$ where φ is an entire function whose values are real along the real axis.

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13.3.3. Give an example of a function that is holomorphic in the open unit disc and has infinitely many roots there.

Solution: For instance, such a function is $\sin\frac{1}{1-z}$ with zeros $1-\frac{1}{k\pi}$.

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14.3.12. Let $D = \{z\in\mathbb{C} : |z| < 1\}$ be the complex unit disc and let $0 < a < 1$ be a real number. Suppose that $f : D\to\mathbb{C}$ is a holomorphic function such that $f(a) = 1$ and $f(-a) = -1$.

(a) Prove that

$$\sup_{z\in D}\left|f(z)\right| \ge \frac{1}{a}.$$

(b) Prove that if f has no root, then

$$\sup_{z\in D}\left|f(z)\right| \ge \exp\left(\frac{1-a^2}{4a}\pi\right).$$

(Schweitzer competition, 2012)

Solution: (a) Let $g(z) = \dfrac{f(z) - f(-z)}{2z}$ for $z \neq 0$ and let $g(0) = f'(0)$. This is a holomorphic function too, satisfying $g(a) = \frac{1-(-1)}{2a} = \frac{1}{a}$. For $a < r < 1$, by the triangle inequality and the maximum principle we have

$$\sup_{z\in D} |f(z)| \geq \max_{|z|=r} |f(z)| \geq r \cdot \max_{|z|=r} \frac{|f(z)| + |f(-z)|}{2r} \geq$$
$$\geq r \cdot \max_{|z|=r} |g(z)| \geq r \cdot |g(a)| = \frac{r}{a}.$$

From $r \to 1 - 0$ the statement follows.

(b) Let $M = \sup\limits_{z\in D} |f(z)|$. Since f is not constant, $|f| < M$ everywhere in D. In particular, from $f(a) = 1$ we can see that $M > 1$.

The function f is non-zero on the simply connected set D, so it has a logarithm; there exists a holomorphic function $g(z) : D \to \mathbb{C}$ such that $f(z) = \exp g(z)$. Without loss of generality we can assume that $g(a) = 0$. From $f(-a) = -1$ we get $g(-a) = k\pi i$ with some odd integer k, and from $|f| < M$ we get $\operatorname{Re} g < \log M$. Denote by H the half-plane $\operatorname{Re} z < \log M$. Hence g is a $D \to H$ function.

Define the linear fractional transformations

$$\varphi : D \to D, \quad \varphi(z) = \frac{z + a}{1 + az}, \quad \varphi^{-1}(z) = \frac{z - a}{1 - az}$$

and

$$\psi : H \to D, \quad \psi(z) = \frac{z}{2 \log M - z}.$$

Consider the $D \to D$ function $h = \psi \circ g \circ \varphi$. Since $\varphi(0) = a$, $g(a) = 0$ and $\psi(0) = 0$, we have $h(0) = 0$. Schwarz's lemma, applied to h and the point $\varphi^{-1}(-a) = \frac{-2a}{1+a^2}$ gives us $\left|h\left(\frac{-2a}{1+a^2}\right)\right| \leq \frac{2a}{1+a^2}$, so

$$\frac{2a}{1 + a^2} \geq \left|h(\varphi^{-1}(-a))\right| = |\psi(g(-a))| = \left|\frac{k\pi i}{2 \log M - k\pi i}\right| = \frac{1}{\sqrt{\left(\frac{2 \log M}{|k|\pi}\right)^2 + 1}}$$

$$\log M \geq \frac{|k|\pi}{2} \sqrt{\left(\frac{1 + a^2}{2a}\right)^2 - 1} = \frac{|k|\pi}{2} \cdot \frac{1 - a^2}{2a} \geq \frac{1 - a^2}{4a} \pi.$$

Remark: The estimates in the problem statement are sharp. For example, we have equality for $f(z) = \dfrac{z}{a}$ in part (a), and for $f(z) = -i \exp\left(\dfrac{iz - a^2}{iz + 1} \cdot \dfrac{\pi}{2a}\right)$ in part (b).

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